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Generalized volume conjecture and the A-polynomials: The Neumann–Zagier potential function as a classical limit of the partition function

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Abstract

We introduce and study the partition function $Z_\gamma(\mathcal{M})$ for the cusped hyperbolic 3-manifold $\mathcal{M}$. We construct formally this partition function based on an oriented ideal triangulation of $\mathcal{M}$ by assigning to each tetrahedron the quantum dilogarithm function, which is introduced by Faddeev in his studies of the modular double of the quantum group. Following Thurston and Neumann–Zagier, we deform a complete hyperbolic structure of $\mathcal{M}$, and we define the partition function $Z_\gamma(\mathcal{M}_u)$ correspondingly. This function is shown to give the Neumann–Zagier potential function in the classical limit $\gamma \to 0$, and the A-polynomial can be derived from the potential function. We explain our construction by taking examples of 3-manifolds such as complements of hyperbolic knots and a punctured torus bundle over the circle.

Keywords: Quantum invariant; Chern–Simons theory; Jones polynomial; Hyperbolic manifold; A-polynomial; Neumann–Zagier function; Quantum dilogarithm function

1. Introduction

Since the quantum invariant of knots/links and 3-manifolds as a generalization of the Jones polynomial [29] was constructed by Witten [61] by use of the Chern–Simons path integral, studies on quantum invariants have been much developed. Recently geometrical interpretations of the quantum invariants have received interest since Kashaev observed an intriguing relationship [33] between the hyperbolic volume and his knot invariant, which is later identified with a specific value of the N-colored Jones polynomial $J_{\mathcal{K}}\left(N; e^{2\pi i/N}\right)$ [45] (here the N-colored Jones polynomial is normalized to be $J_{\text{unknot}}(N; q) = 1$). In particular, the hyperbolic volume of the knot complement $S^3 \setminus \mathcal{K}$ is conjectured to dominate the asymptotics of the invariant $J_{\mathcal{K}}\left(N; e^{2\pi i/N}\right)$ in the large-N limit $N \to \infty$,

$$\lim_{N\to\infty} \frac{2\pi}{N} \log \left| J_{\mathcal{K}}\left(N; e^{2\pi i/N}\right)\right| = \mathrm{Vol}(S^3 \setminus \mathcal{K}). \tag{1.1}$$

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This "volume conjecture" is generalized to other values (near the Nth root of unity) of the N-colored Jones polynomial [22] (see also Refs. [43,44]), and a relationship with the A-polynomial is conjectured; when we define b by

$$b = -\frac{\mathrm{d}}{\mathrm{d}a} \lim_{\substack{N,k\to\infty \\ N/k=a}} \frac{1}{k} \log J_{\mathcal{K}}\left(N; \mathrm{e}^{2\pi\mathrm{i}/k}\right) \tag{1.2}$$

the pair $\left(\mathrm{e}^{b}, -\mathrm{e}^{\mathrm{i}a}\right)$ is a zero locus of the A-polynomial for the knot $\mathcal{K}$. This is checked partially numerically for twist knots [23].

The A-polynomial is originally defined as an algebraic curve of eigenvalues of the $SL(2;\mathbb{C})$ representation of the boundary torus of the knot $\mathcal{K}$ [12] (see also Ref. [13]). This can be computed from the triangulation of the knot complement $\mathcal{M} = S^3 \setminus \mathcal{K}$ into ideal tetrahedra in the hyperbolic space $\mathbb{H}^3$ once the fundamental group has an irreducible representation ρ into $PSL(2;\mathbb{C})$, which is identified with the orientation-preserving isometries of $\mathbb{H}^3$;

$$\rho : \pi_1(\mathcal{M}) \to PSL(2;\mathbb{C}) \simeq \mathrm{Isom}^+(\mathbb{H}^3).$$

Up to conjugation, the meridian μ and the longitude λ of the boundary torus of $\mathcal{K}$ have

$$\rho(\mu) = \begin{pmatrix} m & * \\ 0 & 1/m \end{pmatrix}$$
$$\rho(\lambda) = \begin{pmatrix} \ell & * \\ 0 & 1/\ell \end{pmatrix}.$$

Another geometrical aspect of the N-colored Jones polynomial $J_{\mathcal{K}}(N;q)$ as a relationship with the A-polynomial is proposed as an "AJ conjecture" [21]; the recursion relation of the colored Jones polynomial with respect to N is conjectured to be related to the A-polynomial $A_{\mathcal{K}}(\ell, m)$. This conjecture is proved for the torus knots [27] and the 2-bridge knots [39].

In this paper, we shall introduce a partition function for the cusped hyperbolic manifold $\mathcal{M}$ and its deformation $\mathcal{M}_u$ à la Thurston [57] following a method of Refs. [23–25], and study a classical limit thereof. Based on a triangulation of the cusped 3-manifold $\mathcal{M}$, we define the partition function $Z_\gamma(\mathcal{M}_u)$ by assigning Faddeev's quantum dilogarithm function to each oriented ideal tetrahedron. Originally, Kashaev introduced his invariant $J_{\mathcal{K}}\left(N; \mathrm{e}^{2\pi\mathrm{i}/N}\right)$ for triangulated 3-manifolds, although the R-matrix construction is developed subsequently [32]. He studied Faddeev's quantum dilogarithm function when q is a root of unity [17,30] (see also Ref. [4]), and assigning the quantum dilogarithm function to the ideal tetrahedron, he defined invariants [32,31]. In this sense, our function $Z_\gamma(\mathcal{M})$ for $\mathcal{M}$ with a complete hyperbolic structure can be regarded as a non-compact $U_q(sl(2;\mathbb{R}))$ analogue of the Kashaev invariant. Though we do not know the true content of our partition function and we have no rigorous proof on convergence, the asymptotic behavior of $Z_\gamma(S^3 \setminus \mathcal{K})$ in the limit $\gamma \to 0$ is expected to coincide with that of the Kashaev invariant $J_{\mathcal{K}}\left(N; \mathrm{e}^{2\pi\mathrm{i}/N}\right)$ in the limit $N \to \infty$, as will be discussed below.

One of the advantages of our partition function is a new algorithm to compute the A-polynomial. Once the triangulation of the cusped 3-manifold is given, we can obtain the Neumann–Zagier potential function from a classical limit of the integral expression of $Z_\gamma(\mathcal{M}_u)$, which can be computed by assigning an operator to each oriented ideal tetrahedron. Then the Neumann–Zagier function leads to the A-polynomial straightforwardly, although there remain computational difficulties in the elimination of variables. Our construction works not only for complements of hyperbolic knots, but also for once-punctured torus bundles over the circle. As far as we know, there is no literature concerning the explicit computation of quantum invariant for once-punctured torus bundles over the circle, though it may be done along the lines of Refs. [2,3]. We remark that the potential function associated to hyperbolic structures of knot complements is studied in Ref. [63], based on Kashaev invariant (see also Ref. [46]).

The construction of the partition function $Z_\gamma(\mathcal{M})$ is parallel to that in Refs. [2,3]. Therein, a family of matrix dilogarithms [17] is used for each tetrahedron, unlike non-compact quantum dilogarithm function used here. A non-compact version enables us to define a partition function for the deformed manifold (not hyperbolic complete) M_u based on a geometrical insight, and to regard our function $Z_\gamma(\mathcal{M}_u)$ as a quantum analogue of the Neumann–Zagier potential function.

This paper is organized as follows. In Section 2, we recall definitions of the quantum dilogarithm function, and we discuss the properties of this function. In Section 3, we shall reveal that the three-dimensional hyperbolic geometry naturally arises from the quantum dilogarithm function in the classical limit $\gamma \to 0$, as was clarified in Ref. [24]. In other words, the quantum dilogarithm denotes a γ-deformation of the hyperbolic geometry. Then we define the partition function $Z_\gamma(\mathcal{M}_u)$ for a deformation of complete hyperbolic cusped 3-manifold $\mathcal{M}$. Based on a triangulation of $\mathcal{M}$, and on the fact that the quantum dilogarithm denotes a quantum deformation of the hyperbolic ideal tetrahedron, we construct the partition function by assigning a quantum dilogarithm function to an oriented ideal tetrahedron. We discuss that the Neumann–Zagier potential function appears as a classical limit of $Z_\gamma(\mathcal{M}_u)$. In Section 4, we take several examples of cusped hyperbolic manifolds, such as complements of hyperbolic knots and punctured torus bundles over the circle, and explain our assertion in detail. We shall also give a list for other manifolds in Appendix. The last section is devoted to conclusions and discussions.

2. Quantum dilogarithm function

We define a function $\Phi_\gamma(\varphi)$ by an integral form following Ref. [16]. We set $\gamma \in \mathbb{R}$, and for $|\mathrm{Im}\,\varphi| < \pi$, we define

$$\Phi_\gamma(\varphi) = \exp\left(\int_{\mathbb{R}+\mathrm{i}0} \frac{\mathrm{e}^{-\mathrm{i}\varphi x}}{4\sinh(\gamma x)\sinh(\pi x)}\frac{\mathrm{d}x}{x}\right). \tag{2.1}$$

The Faddeev integral (2.1), which we call the quantum dilogarithm function, is also related to the double sine function [35,55,38], the hyperbolic gamma function [53,1] and the quantum exponential function [62]. We see that the integral $\Phi_\gamma(\varphi)$ has a duality,

$$\Phi_{\frac{\pi^2}{\gamma}}(\varphi) = \Phi_\gamma\left(\frac{\gamma}{\pi}\varphi\right) \tag{2.2}$$

and that it satisfies the inversion relation,

$$\Phi_\gamma(\varphi)\cdot\Phi_\gamma(-\varphi) = \exp\left(-\frac{1}{2\mathrm{i}\gamma}\left(\frac{\varphi^2}{2} + \frac{\pi^2+\gamma^2}{6}\right)\right). \tag{2.3}$$

The Faddeev integral satisfies the difference equations

$$\frac{\Phi_\gamma(\varphi+\mathrm{i}\gamma)}{\Phi_\gamma(\varphi-\mathrm{i}\gamma)} = \frac{1}{1+\mathrm{e}^{\varphi}} \tag{2.4a}$$

$$\frac{\Phi_\gamma(\varphi+\mathrm{i}\pi)}{\Phi_\gamma(\varphi-\mathrm{i}\pi)} = \frac{1}{1+\mathrm{e}^{\frac{\pi}{\gamma}\varphi}}. \tag{2.4b}$$

Due to these relations, the integral $\Phi_\gamma(\varphi)$ defined in (2.1) is analytically continued to $\varphi \in \mathbb{C}$, and we see that

$$\text{zeros of } \left(\Phi_\gamma(\varphi)\right)^{\pm 1} = \left\{\varphi = \mp\mathrm{i}\left((2m+1)\gamma + (2n+1)\pi\right) \mid m, n \in \mathbb{Z}_{\geq 0}\right\}. \tag{2.5}$$

The most important property of the Faddeev integral is that it fulfills the pentagon identity [17,16]

$$\Phi_\gamma(\hat{p})\Phi_\gamma(\hat{q}) = \Phi_\gamma(\hat{q})\Phi_\gamma(\hat{p}+\hat{q})\Phi_\gamma(\hat{p}) \tag{2.6}$$

where $\hat{p}$ and $\hat{q}$ are the canonically conjugate operators satisfying the Heisenberg commutation relation,

$$[\hat{p}, \hat{q}] = \hat{p}\hat{q} - \hat{q}\hat{p} = -2\mathrm{i}\gamma. \tag{2.7}$$

By this commuting relation, we call a limit $\gamma \to 0$ a classical limit. Hereafter, we use $\mathbf{V}$ as the momentum space $|p\rangle$ with $p \in \mathbb{R}$ which is an eigenstate of the momentum operator;

$$\hat{p}|p\rangle = p|p\rangle. \tag{2.8}$$

A reason for the *quantum* dilogarithm function is revealed when we take a *classical* limit $\gamma \to 0$. In this limit, the Faddeev integral reduces to

$$\Phi_\gamma(\varphi) \sim \exp\left(\frac{1}{2\mathrm{i}\gamma}Li_2(-\mathrm{e}^{\varphi})\right). \tag{2.9}$$

Here, $Li_2(x)$ denotes the Euler dilogarithm function defined by (see, *e.g.*, Refs. [40,36])

$$Li_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \tag{2.10}$$

where $|x| \leq 1$. For $x \in \mathbb{C}$, we use the integral form

$$Li_2(x) = -\int_0^x \log(1-s) \frac{\mathrm{d}s}{s}$$

where the branch of $\log(1-s)$ is on $\mathbb{C} \setminus [1, \infty)$, for which $\log(1-0) = 0$. See that the inversion relation (2.3) of the quantum dilogarithm function $\Phi_\gamma(\varphi)$ gives that of the Euler dilogarithm function as

$$Li_2(-\mathrm{e}^x) + Li_2(-\mathrm{e}^{-x}) + \frac{x^2}{2} + \frac{\pi^2}{6} = 0.$$

The Fourier transformation of the Faddeev integral can be computed as follows [62,18,8,35];

$$\frac{1}{\sqrt{4\pi\gamma}} \int_{\mathbb{R}} \mathrm{d}y\ \Phi_\gamma(y) \mathrm{e}^{\frac{1}{2\mathrm{i}\gamma} xy} = \Phi_\gamma(-x + \mathrm{i}\pi + \mathrm{i}\gamma) \mathrm{e}^{\frac{1}{2\mathrm{i}\gamma}\left(\frac{x^2}{2} - \frac{1}{2}\pi\gamma - \frac{\pi^2+\gamma^2}{6}\right)} \tag{2.11}$$

$$\frac{1}{\sqrt{4\pi\gamma}} \int_{\mathbb{R}} \mathrm{d}y \frac{1}{\Phi_\gamma(y)} \mathrm{e}^{\frac{1}{2\mathrm{i}\gamma} xy} = \frac{1}{\Phi_\gamma(x - \mathrm{i}\pi - \mathrm{i}\gamma)} \mathrm{e}^{-\frac{1}{2\mathrm{i}\gamma}\left(\frac{x^2}{2} - \frac{1}{2}\pi\gamma - \frac{\pi^2+\gamma^2}{6}\right)}. \tag{2.12}$$

To see a relationship between the integral $\Phi_\gamma(\varphi)$ and geometry, we define the S-operator acting on $\mathbf{V} \otimes \mathbf{V}$ by

$$S_{1,2} = \mathrm{e}^{\frac{1}{2\mathrm{i}\gamma} \hat{q}_1 \hat{p}_2} \Phi_\gamma(\hat{p}_1 + \hat{q}_2 - \hat{p}_2). \tag{2.13}$$

Here the Heisenberg operators $\hat{p}_j$ and $\hat{q}_j$ act on the jth vector space of $\mathbf{V} \otimes \mathbf{V}$, *i.e.* $\hat{p}_1 = \hat{p} \otimes 1$, $\hat{p}_2 = 1 \otimes \hat{p}$, and so on. Then, the pentagon identity (2.6) can be rewritten in a compact form:

$$S_{2,3} S_{1,2} = S_{1,2} S_{1,3} S_{2,3} \tag{2.14}$$

where $S_{j,k}$ acts as $S_{1,2}$ on the j and kth spaces of $\mathbf{V} \otimes \mathbf{V} \otimes \mathbf{V}$ and as the identity on the rest. Matrix elements can be computed by use of (2.11) and (2.12);

$$\langle p_1, p_2 | S_{1,2} |\ p'_1, p'_2 \rangle = \frac{1}{\sqrt{4\pi\gamma}} \delta(p_1 + p_2 - p'_1) \cdot \Phi_\gamma(p'_2 - p_2 + \mathrm{i}\pi + \mathrm{i}\gamma) \mathrm{e}^{\frac{1}{2\mathrm{i}\gamma}\left(-\frac{\pi^2+\gamma^2}{6} - \frac{\gamma\pi}{2} + p_1(p'_2 - p_2)\right)} \tag{2.15a}$$

$$\langle p_1, p_2 | S^{-1}_{1,2} | p'_1, p'_2 \rangle = \frac{1}{\sqrt{4\pi\gamma}} \delta(p_1 - p'_1 - p'_2) \frac{1}{\Phi_\gamma(p_2 - p'_2 - \mathrm{i}\pi - \mathrm{i}\gamma)} \mathrm{e}^{\frac{1}{2\mathrm{i}\gamma}\left(\frac{\pi^2+\gamma^2}{6} + \frac{\gamma\pi}{2} - p'_1(p_2 - p'_2)\right)}. \tag{2.15b}$$

In the classical limit $\gamma \to 0$, we find by use of (2.9) that the S-operators (2.15) reduce to

$$\langle p_1, p_2 | S_{1,2} | p'_1, p'_2 \rangle \sim \delta(p_1 + p_2 - p'_1) \cdot \exp\left(-\frac{1}{2\mathrm{i}\gamma} V(p'_2 - p_2, p_1)\right) \tag{2.16a}$$

$$\langle p_1, p_2 | S^{-1}_{1,2} |\ p'_1, p'_2 \rangle \sim \delta(p_1 - p'_1 - p'_2) \cdot \exp\left(\frac{1}{2\mathrm{i}\gamma} V(p_2 - p'_2, p'_1)\right) \tag{2.16b}$$

where we have defined the function $V(x, y)$ by

$$V(x, y) = \frac{\pi^2}{6} - Li_2(\mathrm{e}^x) - xy. \tag{2.17}$$

We see that the function $V(x, y)$ satisfies the partial differential equations:

$$V(x, y) = L(1 - \mathrm{e}^x) + \frac{1}{2}\left(x \frac{\partial V(x, y)}{\partial x} + y \frac{\partial V(x, y)}{\partial y}\right) \tag{2.18}$$

$$\operatorname{Im} V(x,y) = D(1-\mathrm{e}^x) + \log|\mathrm{e}^x| \cdot \operatorname{Im}\left(\frac{\partial}{\partial x} V(x,y)\right) + \log|\mathrm{e}^y| \cdot \operatorname{Im}\left(\frac{\partial}{\partial y} V(x,y)\right). \tag{2.19}$$

Here, the Rogers dilogarithm $L(z)$ and the Bloch–Wigner function $D(z)$ are respectively defined in terms of the Euler dilogarithm function (2.10) by

$$L(z) = Li_2(z) + \frac{1}{2}\log z \log(1-z) \tag{2.20}$$

$$D(z) = \operatorname{Im} Li_2(z) + \arg(1-z)\cdot\log|z| \tag{2.21}$$

both of which fulfill the pentagon identity (see, *e.g.*, Ref. [40]);

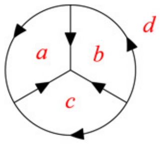
$$L(z) - L(w) + L\left(\frac{w}{z}\right) - L\left(\frac{1-z^{-1}}{1-w^{-1}}\right) + L\left(\frac{1-z}{1-w}\right) = \frac{\pi^2}{6} \tag{2.22}$$

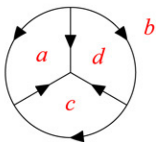
$$D(z) - D(w) + D\left(\frac{w}{z}\right) - D\left(\frac{1-z^{-1}}{1-w^{-1}}\right) + D\left(\frac{1-z}{1-w}\right) = 0. \tag{2.23}$$

3. Partition function and potential function

3.1. The S-operator and hyperbolic ideal tetrahedron

The natural interpretation of the pentagon identity (2.14) is the $2 \leftrightarrow 3$ Pachner move (bistellar move along face/edge). In our case, the matrix elements of the S-operator have four indices, and we can assign an (oriented) tetrahedron to each S-operator as follows (see Refs. [10,34,7] also Ref. [11] for another interpretation as a quantization of the Teichmüller theory);

$\langle a, b \mid S \mid c, d \rangle =$ 

$\langle a, b \mid S^{-1} \mid c, d \rangle =$ 

Here, we assign momenta $a, b, c, d \in \mathbf{V}$ to each face, and we interpret an integration with respect to p, $\int_{\mathbb{R}} \mathrm{d}p |p\rangle\langle p|$, as gluing two faces together. As no faces have loops, there is no ambiguity in gluing faces together once any two faces to be glued are fixed. Furthermore, one finds that in-states $|p\rangle$ can glue only to out-states $\langle p|$, and that we cannot glue two in-states or out-states together.

By this interpretation of the S-operators with the oriented tetrahedra, the pentagon identity (2.14) is identified with the Pachner move as is depicted in Fig. 1; oriented polytope with 5 vertices can be decomposed into 2 tetrahedra with a face in common, or into 3 tetrahedra with an edge in common. Further to identify the oriented tetrahedra as hyperbolic *ideal* tetrahedra whose vertices are at infinity, we study an explicit form of the matrix elements of the right hand side of the pentagon identity (2.14), which is read in $\gamma \to 0$ as

$$\begin{aligned}
&\iiint \mathrm{d}y\mathrm{d}z\mathrm{d}w \langle p_1, p_2|S|y, z\rangle\langle y, p_3|S|p_1', w\rangle\langle z, w|S|p_2', p_3'\rangle \\
&\quad \sim \delta(p_1+p_2+p_3-p_1') \int \mathrm{d}z \exp\left[\frac{1}{2\mathrm{i}\gamma}\left(-\frac{\pi^2}{2} + Li_2(\mathrm{e}^{z-p_2}) + Li_2(\mathrm{e}^{p_2'-p_3-z})\right.\right. \\
&\qquad \left.\left. + \ Li_2(\mathrm{e}^{p_3'-p_2'+z}) + z\left(-p_2+p_3'-p_2'+z\right) - p_1 p_2 + \left(p_2'-p_3\right)(p_1+p_2)\right)\right].
\end{aligned}$$

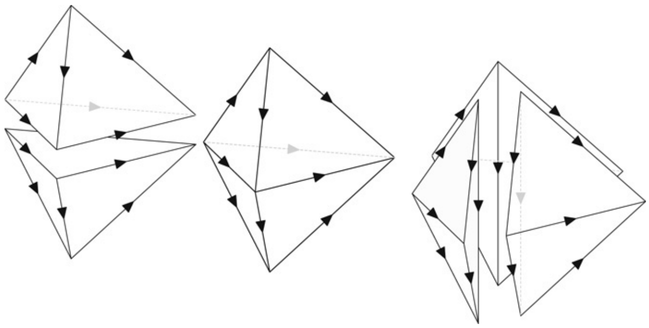


Fig. 1. Pentagon identity (2.14) is interpreted as the 2 ↔ 3 Pachner move.

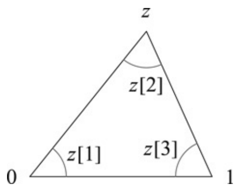
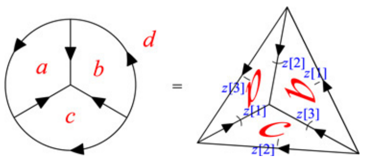


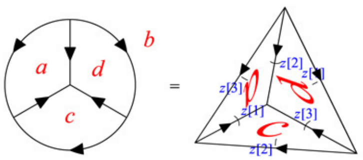
Fig. 2. Triangle with vertices 0, 1, and z in $\mathbb{C}$. Here, we set $z[1] = z$, $z[2] = 1 - \frac{1}{z}$, and $z[3] = \frac{1}{1-z}$.

We may apply the saddle point method in the above integral, and we obtain

$$\left(1 - \mathrm{e}^{p_2' - z - p_3}\right)^{-1} \left(1 - \mathrm{e}^{p_2 - z}\right) \left(1 - \mathrm{e}^{p_2' - z - p_3'}\right) = 1.$$

Our assertion in Ref. [24] is that this condition exactly coincides with the hyperbolic consistency condition in gluing three tetrahedra around a common edge (see Fig. 1), once the tetrahedra assigned to the S-operators $\langle a, b|S^{\pm 1}|c, d\rangle$ are regarded as the hyperbolic *ideal* tetrahedra with moduli e^{d-b} as follows;

$\langle a, b \mid S \mid c, d\rangle =$  (3.1)

$\langle a, b \mid S^{-1} \mid c, d\rangle =$  . (3.2)

The rightmost figures represent oriented *ideal* hyperbolic tetrahedra, and $z[a]$ is the dihedral angle;

$$z[1] = z = \mathrm{e}^{d-b}$$
$$z[2] = 1 - \frac{1}{z}$$
$$z[3] = \frac{1}{1-z}.$$

Here, $z = \mathrm{e}^{d-b}$ is the modulus, and the cross section by the horosphere is similar to the triangle in $\mathbb{C}$ with vertices 0, 1, and z (see Fig. 2), and we have $z[1]z[2]z[3] = -1$. See that the opposite edges of tetrahedra have the same dihedral angles.

Coincidence between the saddle point equations and the hyperbolic consistency conditions can be seen for any other orientations of 2 ↔ 3 Pachner moves, and for any other type of gluing of ideal tetrahedra around common edges. See Ref. [24] for detail. This supports our identifications of the S-operators with hyperbolic *ideal* tetrahedra. Another reason for the *ideal* tetrahedra is a relation with the volume. As can be seen from (2.16), (2.18) and (2.19), the asymptotics of the imaginary part of the S-operator in the classical limit $\gamma \to 0$ is dominated by the Bloch–Wigner function, because the extra terms in (2.18) and (2.19) vanish due to the hyperbolic consistency conditions [24]. Indeed the Bloch–Wigner function $D(z)$ denotes the hyperbolic volume of the ideal tetrahedron $\Delta(z)$ with modulus z [5,42,57],

$$\begin{aligned}\mathrm{Vol}\,(\Delta(z)) &= D(z)\\ &= Л\,(\arg(z[1])) + Л\,(\arg(z[2])) + Л\,(\arg(z[3]))\end{aligned} \tag{3.3}$$

where $Л(\theta)$ is the Lobachevsky function defined by

$$Л(\theta) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(2n\theta)}{n^2}.$$

In view of these facts, we can assign a hyperbolic *ideal* oriented tetrahedron to the S-operator (2.15) as in (3.1) and (3.2) with modulus e^{d-b}. As a result, we see that the S-operator (2.15) denotes a quantum deformation of the hyperbolic volume.

This identification differs from the $6j$ symbol as a solution of the pentagon identity which was used to define the topological gravity [51]. The Ponzano–Regge partition function diverges in general, though it still enjoys a geometrical interpretation (see *e.g.*, Ref. [52]). The quantization thereof is the Turaev–Viro state sum invariant [59], in which the quantum $6j$-symbol is used [37]. A relationship between the quantum $6j$-symbol and the hyperbolic tetrahedron is suggested in Ref. [47].

3.2. Partition function of a cusped 3-manifold

Any hyperbolic cusped 3-manifold $\mathcal{M}$ (for simplicity we assume that the number of cusp is one in this paper) can be ideally triangulated, and it is constructed from a finite number of the oriented ideal tetrahedra in (3.1) and (3.2). Other types of oriented ideal tetrahedra, such as one that has a face with a loop, are prohibited. So our triangulation sometimes differs from the canonical triangulation used in computer programs, such as SnapPea [60], Knotscape [28], and Snap [14].

Once a triangulation is given and we know how to glue faces together, we can naturally define the partition function for a hyperbolic cusped 3-manifold $\mathcal{M}$ based on the S-operator by

$$Z_\gamma(\mathcal{M}) = \int_{\mathbb{R}} \mathrm{d}p\,\delta_C(p)\delta_G(p)\prod_{i=1}^{M}\left\langle p_{2i-1}^{(-)}, p_{2i}^{(-)}\,|S^{\varepsilon_i}|\,p_{2i-1}^{(+)}, p_{2i}^{(+)}\right\rangle \tag{3.4}$$

where p denotes a set of variables $(p_1^{(\pm)}, p_2^{(\pm)}, \ldots, p_{2M}^{(\pm)})$, and $\varepsilon_i = \pm 1$, depending on an orientation of tetrahedron. We set M as the number of ideal tetrahedra. The condition $\delta_G(p)$ determines how to glue faces together ("G" stands for "gluing"). Every face with same momentum has to be glued together, and the fact that in-states can be glued only to out-states indicates that the gluing condition $\delta_G(p)$ is a product of $\delta(p_j^{(-)} - p_k^{(+)})$ for some j and k.

We need another geometrical condition $\delta_C(p)$ to define a partition function in addition to a way to glue faces ("C" stands for "completeness") [26]. We can draw a developing map from an ideal triangulation of a 3-manifold, and we need to read off a hyperbolic complete condition. This condition can be written as a constraint for p by identifications (3.1) and (3.2). By construction, the partition function $Z_\gamma(\mathcal{M})$ is invariant under the Pachner move (Fig. 1) with any orientations, and it is essentially the invariant constructed in Ref. [24].

As was studied by Thurston [58], a *deformation* of the hyperbolic structure on a manifold $\mathcal{M}$ can be holomorphically parametrized by a parameter u in a neighborhood of completeness condition $u = 0$. The parameter u is the logarithm of the eigenvalue of the meridian by the holonomy representation, and we set

$$m = \mathrm{e}^u. \tag{3.5}$$

Correspondingly we denote such a manifold by $\mathcal{M}_u$, which is no more complete, and define the partition function by

$$Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p \delta_C(p; u) \delta_G(p) \prod_{i=1}^{M} \left\langle p_{2i-1}^{(-)}, p_{2i}^{(-)} | S^{\varepsilon_i} | p_{2i-1}^{(+)}, p_{2i}^{(+)} \right\rangle. \tag{3.6}$$

Here, the condition $\delta_C(p; u)$ follows from the fact that the meridian has a holonomy (3.5). The partition function (3.4) for $\mathcal{M}$ with a complete hyperbolic structure follows from (3.6) by assuming a completeness $u = 0$ of the hyperbolic structure of $\mathcal{M}_u$;

$$Z_\gamma(\mathcal{M}) = Z_\gamma\,(\mathcal{M}_{u=0}) \tag{3.7}$$

because

$$\delta_C(p) = \delta_C(p; u = 0). \tag{3.8}$$

Although the integral forms, (3.4) and (3.6), respectively give the partition functions of $\mathcal{M}$ and $\mathcal{M}_u$, their true contents are not obvious. It remains for future studies using mathematically rigorous analyses of our partition functions to disclose these. Nonetheless, it is important that we can evaluate dominating terms of partition functions in the classical limit $\gamma \to 0$ by the saddle point method, and that it can be interpreted geometrically as the hyperbolic ideal triangulations. In this sense, the partition function $Z_\gamma(\mathcal{M}_u)$ may include hyperbolic geometrical information about manifold $\mathcal{M}_u$. Furthermore, we have a new algorithm to compute the A-polynomial of the cusped hyperbolic 3-manifold, which we explain below.

To see the hidden hyperbolic structure of our partition function, we study a classical limit $\gamma \to 0$ of our partition function $Z_\gamma(\mathcal{M}_u)$. When we take a limit $\gamma \to 0$ by use of (2.16), the asymptotics of the partition function defined by (3.6) are dominated by

$$\begin{aligned} Z_\gamma\,(\mathcal{M}_u) &\sim \int_{\mathbb{R}} \mathrm{d}p \delta_C(p; u) \delta_G(p) \left[\prod_{i=1}^{M} \delta\left(p_{2i-1}^{(\varepsilon_i)} - p_{2i-1}^{(-\varepsilon_i)} - p_{2i}^{(-\varepsilon_i)} \right) \right] \\ &\quad \times \exp\left(\frac{\mathrm{i}}{2\gamma} \sum_{i=1}^{M} \varepsilon_i V\left(p_{2i}^{(\varepsilon_i)} - p_{2i}^{(-\varepsilon_i)}, p_{2i-1}^{(-\varepsilon_i)} \right) \right) \\ &= \int_{\mathbb{R}} \mathrm{d}x \exp\left(\frac{1}{2\mathrm{i}\gamma} \Phi_{\mathcal{M}}(x; u) \right). \end{aligned} \tag{3.9}$$

Here in the last equality, following our convention we have re-parametrized variables p with $x = (x_1, x_2, \ldots, x_{M-1})$ after incorporating constraints written in terms of delta functions.

The integral (3.9) could be evaluated by the saddle point method, as we have worked a classical limit $\gamma \to 0$. The saddle point condition for variables x is

$$\frac{\partial}{\partial x_i} \Phi_{\mathcal{M}}(x; u) = 0. \tag{3.10}$$

As was extensively studied in Ref. [24], these conditions coincide with the hyperbolic consistency conditions around the edges when we glue oriented tetrahedra together, *i.e.*, unity is the product of dihedral angles around each edge. By construction, the variable u denotes the meridian of the cusp in this classical limit, and the complete hyperbolic structure is realized by setting $u = 0$. To conclude, the function $\Phi_{\mathcal{M}}(x; u)$ defined by a classical limit (3.9) of the partition function $Z_\gamma(\mathcal{M}_u)$ under constraints (3.10) is nothing but the Neumann–Zagier potential function [50,64].

As a result, the differential of the potential function with respect to the deformation parameter u gives

$$\frac{\partial}{\partial u} \Phi_{\mathcal{M}}(x; u) = 2v \tag{3.11}$$

where v is related to the eigenvalue of the longitude by the holonomy representation

$$\ell = \mathrm{e}^{v}. \tag{3.12}$$

Variables x_i can be solved from the hyperbolic consistency equation (3.10) as a function of u, and we can regard the potential function $\Phi_{\mathcal{M}}(x;u)$ as a function of u; $\Phi_{\mathcal{M}}(u) = \Phi_{\mathcal{M}}(x;u)|_{(3.10)}$. Then we can rewrite the (3.11) as

$$\frac{\mathrm{d}}{\mathrm{d}u} \lim_{\gamma\to 0} \mathrm{i}\gamma \log Z_\gamma(\mathcal{M}_u) = v. \tag{3.13}$$

We note that our variables (u, v) differ from those in Ref. [50]; when we denote their variables as $(u_{\mathrm{NZ}}, v_{\mathrm{NZ}})$, we have

$$(u, v) = \left(\frac{u_{\mathrm{NZ}}}{2}, \frac{v_{\mathrm{NZ}}}{2} + \pi\mathrm{i}\right). \tag{3.14}$$

Hereafter, we also use the function $V_{\mathcal{M}}(x;m)$ defined by

$$V_{\mathcal{M}}(x_1, x_2, \ldots; m) = \Phi_{\mathcal{M}}(\log x_1, \log x_2, \ldots; u = \log m). \tag{3.15}$$

As seen from (2.18) and (2.19), the potential function $\Phi_{\mathcal{M}}(x;u)$ under saddle point conditions (3.10) becomes a sum of the Rogers dilogarithm functions. We recall here the Bloch invariant studied in Refs. [49,48,15]. The Bloch invariant $\beta(\mathcal{M})$ is defined for a finite volume hyperbolic 3-manifold $\mathcal{M}$ as

$$\beta(\mathcal{M}) = \sum_{i=1}^{M} [z_i] \tag{3.16}$$

where $[z]$ satisfies the Bloch group

$$[z] - [w] + \left[\frac{w}{z}\right] - \left[\frac{1-z^{-1}}{1-w^{-1}}\right] + \left[\frac{1-z}{1-w}\right] = 0. \tag{3.17}$$

The Bloch regulator map ρ gives [49]

$$\rho(\beta(\mathcal{M})) = \mathrm{Vol}(\mathcal{M}) + \mathrm{i}\,\mathrm{CS}(\mathcal{M}) \tag{3.18}$$

where CS denotes the Chern–Simons invariant defined modulo π^2 (see Ref. [41] for a definition of the Chern–Simons invariant for the case of cusped manifolds). As identities (2.18) and (2.19) show that the S-operator reduces to the Rogers dilogarithm function (or the Bloch–Wigner function) in the saddle point, and that they satisfy the pentagon identities (2.22) and (2.23), we can interpret our partition function $Z_\gamma(\mathcal{M})$ as a quantization of the Bloch invariant.

Generally, we have many saddle points as algebraic solutions of a set of equations (3.10). Among them, a solution which has the largest absolute value dominates the asymptotics of the partition function $Z_\gamma(\mathcal{M})$. Combining this with the fact that our partition function $Z_\gamma(\mathcal{M})$ may be regarded as a quantization of the Bloch invariant, we should have

$$\lim_{\gamma\to 0} 2\gamma \log\left(Z_\gamma(\mathcal{M})\right) = \mathrm{Vol}(\mathcal{M}) + \mathrm{i}\,\mathrm{CS}(\mathcal{M}) \tag{3.19}$$

as a variant of the volume conjecture [24]. However, there still remains an ambiguity of branch in complex plane in an actual computation.

The Neumann–Zagier potential function $\Phi_{\mathcal{M}}(x;u)$ has much information on the geometry of manifolds. One of the properties is a relationship with the A-polynomial defined in Ref. [12]. When the manifold $\mathcal{M}$ is a complement of knot $S^3 \setminus \mathcal{K}$, the A-polynomial $A_{\mathcal{K}}(\ell, m)$ of $\mathcal{K}$ as an algebraic equation of ℓ and m can be given by using the Gröbner base or resultant theory to eliminate x_i from a set of equations, (3.10) and (3.11). So the pair $(\ell, m) = (\mathrm{e}^v, \mathrm{e}^u)$ defined from (3.13) is a zero locus of the A-polynomial $A_{\mathcal{K}}(\ell, m)$. This result should be comparable with the conjecture (1.2), and the partition function $Z_\gamma(\mathcal{M}_u)$ is a *non-compact* generalization of the Jones–Witten invariant.

The A-polynomial has following properties [12,13];

- Polynomial $A_{\mathcal{K}}(\ell, m)$ is an integer polynomial, and it contains only even powers of m.
- Up to powers of ℓ and m, we have
$$A_{\mathcal{K}}(\ell, m) = A_{\mathcal{K}}(1/\ell, 1/m). \tag{3.20}$$
- If $\mathcal{K}$ and $\mathcal{K}'$ are mirror images, then
$$A_{\mathcal{K}}(\ell, m) = A_{\mathcal{K}'}(1/\ell, m). \tag{3.21}$$

- With the additional property that every closed incompressible surface embedded in $S^3\setminus\mathcal{K}$ is parallel to the boundary torus, we have

$$A_{\mathcal{K}}(\ell,\pm 1) = n\,(\ell+1)^{\alpha_+}\,(\ell-1)^{\alpha_-}\,\ell^{\beta} \tag{3.22}$$

with non-zero integer n.
- Slopes of edges of the Newton polygon of $A_{\mathcal{K}}(\ell,m)$ are boundary slopes of $\mathcal{K}$.
- Coefficients of terms in the corners of the Newton polygons of $A_{\mathcal{K}}(\ell,m)$ are ± 1.

Another property of the Neumann–Zagier potential function $\Phi_{\mathcal{M}}(u)$ associated to cusped manifold $\mathcal{M}$ is the volume of the Dehn surgered manifold. The (p,q)-hyperbolic Dehn surgery of $\mathcal{M}$, where (p,q) is a pair of coprime integers, is performed by gluing back a solid torus with cusp of $\mathcal{M}$, where the surgery data satisfy [57]

$$pu+qv=\pi \mathrm{i}. \tag{3.23}$$

Then for the core c of solid torus, we have

$$\mathrm{Length}(c)+\mathrm{i}\mathrm{Torsion}(c) = -2\,(ru+sv) \bmod 2\pi\mathrm{i}. \tag{3.24}$$

where $\begin{pmatrix} p & r\\ q & s\end{pmatrix} \in SL(2,\mathbb{Z})$. We have

$$\mathrm{Im}\,(u\bar{v}) = -\frac{\pi}{2}\mathrm{Length}(c). \tag{3.25}$$

According to Refs. [64,50], we have for the hyperbolic (p,q)-Dehn surgered manifold $\mathcal{M}_{(p,q)}$:

$$\begin{aligned}&\left(\mathrm{Vol}\left(\mathcal{M}_{(p,q)}\right)+\mathrm{iCS}\left(\mathcal{M}_{(p,q)}\right)\right)-(\mathrm{Vol}(\mathcal{M})+\mathrm{iCS}(\mathcal{M}))\\ &\quad = -\frac{\mathrm{i}}{4}\left(\Phi_{\mathcal{M}}(u)-4uv\right)-\frac{\pi}{2}\left(\mathrm{Length}(c)+\mathrm{i}\mathrm{Torsion}(c)\right).\end{aligned} \tag{3.26}$$

which follows from

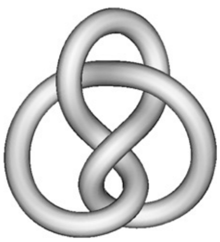
$$Z_\gamma\left(\mathcal{M}_{(p,q)}\right) \sim \int \mathrm{d}u\, \mathrm{e}^{\frac{1}{2\mathrm{i}\gamma}\left(\frac{p}{q}u^2+\frac{2(\pi+\gamma)\mathrm{i}}{q}u\right)} Z_\gamma(\mathcal{M}_u). \tag{3.27}$$

4. Examples

We explain our constructions of the partition function by taking some concrete examples of cusped hyperbolic 3-manifolds.

4.1. *Figure-eight knot* 4_1

We set $\mathcal{K}$ as the figure-eight knot 4_1, which is depicted as



It is well known that the complement of the figure eight knot, $\mathcal{M}=S^3\setminus\mathcal{K}$, is given by two ideal tetrahedra, and the triangulation induces the partition function as

$$Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p\,\delta_C(p;u)\langle p_1,p_2|S|p_3,p_4\rangle\langle p_4,p_3|S^{-1}|p_2,p_1\rangle. \tag{4.1}$$

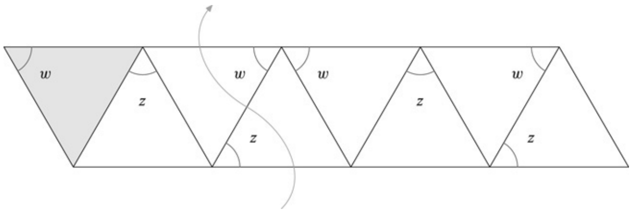


Fig. 3. Developing map of the complement of the figure-eight knot. The gray filled triangle corresponds to top vertex of the tetrahedron (central vertex of circle in (3.1)) with modulus w in the projection of (3.1). A curve denotes a meridian of the cusp.

Modulus of two tetrahedra are given by $w = \mathrm{e}^{p_4 - p_2}$ and $z = \mathrm{e}^{p_1 - p_3}$. The developing map is drawn in Fig. 3, and the meridian is read to be

$$\frac{w}{z} = \mathrm{e}^{-2u}$$

which shows that the condition $\delta_C(p; u)$ is

$$p_4 - p_2 - (p_1 - p_3) = -2u.$$

The complete hyperbolic structure is realized when $u = 0$.

We then obtain

$$Z_\gamma(\mathcal{M}_u) = \frac{1}{4\pi\gamma} \int \mathrm{d}x \, \frac{\Phi_\gamma(x + \mathrm{i}\pi + \mathrm{i}\gamma)}{\Phi_\gamma(-x - 2u - \mathrm{i}\pi - \mathrm{i}\gamma)} \mathrm{e}^{\frac{-1}{2\mathrm{i}\gamma} 4u(u+x)} \tag{4.2}$$

which in the limit $\gamma \to 0$ reduces to

$$Z_\gamma(\mathcal{M}_u) \sim \int \mathrm{d}x \exp\left(\frac{1}{2\mathrm{i}\gamma}\left(Li_2(\mathrm{e}^x) - Li_2(\mathrm{e}^{-x-2u}) - 4u(u+x)\right)\right)$$
$$= \int \mathrm{d}x \exp\left(\frac{1}{2\mathrm{i}\gamma} V_{\mathcal{M}}(\mathrm{e}^x; \mathrm{e}^u)\right). \tag{4.3}$$

Here the potential function is set to be

$$V_{\mathcal{M}}(x; m) = Li_2(x) - Li_2\left(\frac{1}{xm^2}\right) - 4 \log m \log(xm). \tag{4.4}$$

To evaluate the integral (4.3), we may apply the saddle point method, and the condition (3.10) reduces to

$$-\frac{x}{m^2(1-x)(1-m^2x)} = 1. \tag{4.5}$$

From (3.11) the longitude is given by

$$\frac{1}{m^2x(m^2x-1)} = \ell. \tag{4.6}$$

Completeness condition $m = 1$ gives $x = \frac{1 \pm \sqrt{3}\mathrm{i}}{2}$ from (4.5). Substituting this solution for the potential function (4.4), the imaginary part coincides with the hyperbolic volume of the figure-eight knot $\mathrm{Vol}(S^3 \setminus 4_1) = 2.02988\ldots$.

For a deformed manifold $\mathcal{M}_u$, we get an algebraic equation of ℓ and m by eliminating x from (4.5) and (4.6) as

$$A_{\mathcal{M}}(\ell, m) = 0. \tag{4.7}$$

Here $A_{\mathcal{M}}(\ell, m)$ is the A-polynomial for the figure-eight knot;

$$A_{\mathcal{M}}(\ell, m) = -m^4 + \ell(1 - m^2 - 2m^4 - m^6 + m^8) - \ell^2 m^4 \tag{4.8}$$

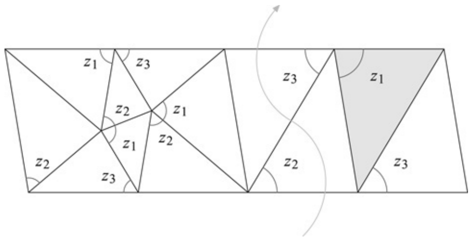


Fig. 4. Developing map of the complement of 5_2. Gray filled triangle denotes top vertex of the tetrahedron with modulus z_1. Meridian is denoted by a gray curve.

which can be expressed as in the Newton polygon as follows:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

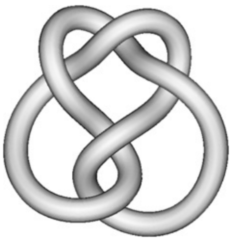
We note that a set of equations (4.5) and (4.6) gives

$$2v = -2\pi \mathrm{i} + 4\sqrt{3}\mathrm{i}u + \frac{16\mathrm{i}}{\sqrt{3}}u^3 + \frac{368\mathrm{i}}{15\sqrt{3}}u^5 + \frac{2848\mathrm{i}}{45\sqrt{3}}u^7 + \cdots$$

which coincides with a result in Ref. [50] under (3.14).

4.2. 5_2

We set $\mathcal{K}$ as the knot 5_2 depicted as



The knot complement $\mathcal{M} = S^3 \setminus \mathcal{K}$ is constructed from three tetrahedra (see, *e.g.*, Ref. [56]), and we obtain the partition function as

$$Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p\, \delta_C(p;u) \langle p_1, p_5 | S^{-1} | p_4, p_3 \rangle \langle p_2, p_4 | S^{-1} | p_6, p_5 \rangle \langle p_3, p_6 | S^{-1} | p_1, p_2 \rangle. \tag{4.9}$$

The moduli of the three tetrahedra are respectively $z_1 = \mathrm{e}^{p_3 - p_5}$, $z_2 = \mathrm{e}^{p_5 - p_4}$, and $z_3 = \mathrm{e}^{p_2 - p_6}$. The developing map is depicted in Fig. 4. The meridian is read to be $\frac{z_3}{z_2}$, and the condition $\delta_C(p;u)$ is

$$p_5 - p_4 + p_6 - p_2 = 2u.$$

We then have

$$Z_\gamma(\mathcal{M}_u) = \frac{1}{(4\pi\gamma)^{3/2}} \iint \mathrm{d}x\,\mathrm{d}y\, \mathrm{e}^{\frac{1}{2\mathrm{i}\gamma}\left(\frac{\pi^2+\gamma^2}{2} + \frac{3}{2}\pi\gamma + (2u-y)(y-x)\right)}$$
$$\times \frac{1}{\Phi_\gamma(-x+y-2u-\mathrm{i}\pi-\mathrm{i}\gamma)\,\Phi_\gamma(-y-2u-\mathrm{i}\pi-\mathrm{i}\gamma)\,\Phi_\gamma(-y-2u-\mathrm{i}\pi-\mathrm{i}\gamma)}. \tag{4.10}$$

In the classical limit $\gamma \to 0$, we obtain

$$Z_\gamma\left(\mathcal{M}_u\right) \sim \iint \mathrm{d}x\mathrm{d}y \exp\left(\frac{1}{2\mathrm{i}\gamma} V_{\mathcal{M}}(\mathrm{e}^x, \mathrm{e}^y; \mathrm{e}^u)\right) \tag{4.11}$$

where the Neumann–Zagier potential function is

$$V_{\mathcal{M}}(x, y; m) = \frac{\pi^2}{2} - Li_2\left(\frac{y}{xm^2}\right) - Li_2\left(\frac{1}{ym^2}\right) - Li_2\left(\frac{1}{y}\right) + \log\left(y/x\right)\log\left(m^2/y\right). \tag{4.12}$$

The integral is evaluated by the saddle point method and saddle point conditions (3.10), which denote hyperbolic consistency conditions, reducing it to

$$\begin{aligned} & m^2 x - y = xy \\ & (m^2 y - 1)(y-1) = m^2(m^2 x - y) \end{aligned} \tag{4.13}$$

and the longitude (3.11) is computed as

$$m^4 y^2 = (m^2 x - y)(m^2 y - 1)\ell. \tag{4.14}$$

In the case of the complete structure $\mathcal{M}$, *i.e.*, $u = 0$, there exists a solution of (4.13), $(x, y) = (-0.877439 + 0.744862\mathrm{i}, 0.78492 + 1.30714\mathrm{i})$, such that the imaginary part of the potential function

$$\operatorname{Im} V_{\mathcal{M}}(x, y; 1) = -D\left(\frac{y}{x}\right) - 2D\left(\frac{1}{y}\right)$$

gives the hyperbolic volume of $\mathcal{M}$; $\mathrm{Vol}(\mathcal{M}) = 2.82812\ldots$. Correspondingly, we have

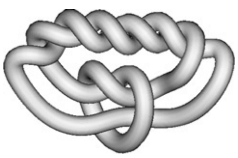
$$\begin{aligned} V_{\mathcal{M}}(x, y; 1) &= \frac{\pi^2}{2} - L\left(\frac{y}{x}\right) - 2L\left(\frac{1}{y}\right) \\ &= 2\pi^2 \cdot 0.153204\ldots + \mathrm{i} \cdot 2.82812\ldots. \end{aligned}$$

To get the A-polynomial, we eliminate variables x and y from a set of equations (4.13) and (4.14). After some algebra, we obtain $A_{\mathcal{M}}(\ell, m) = 0$, where the function $A_{\mathcal{M}}(\ell, m)$ coincides with the A-polynomial for 5_2 given by the following Newton polygon:

$$\begin{pmatrix} -1 & 1 & & \\ & -2 & & \\ & -2 & -1 & \\ & & 1 & \\ & 1 & & \\ & -1 & -2 & \\ & & -2 & \\ & & 1 & -1 \end{pmatrix}.$$

4.3. *Pretzel knot (−2,3,7)*

Let $\mathcal{K}$ be the $(-2, 3, 7)$ Pretzel knot depicted as



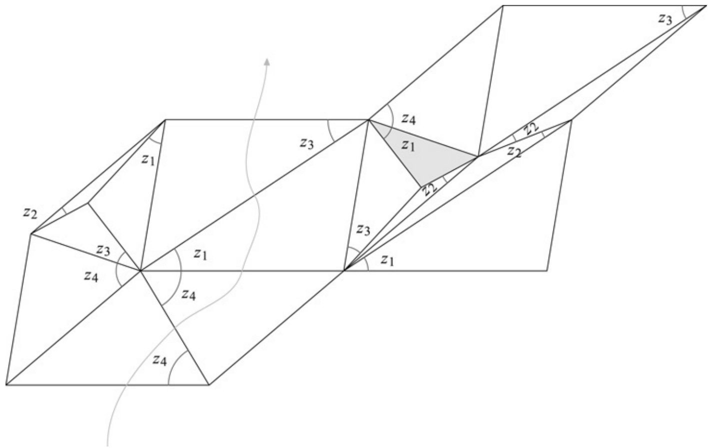


Fig. 5. Developing map of complement of the Pretzel knot (−2,3,7). Horosphere of the top vertex of the tetrahedron with modulus z_1 is depicted by gray triangle. Gray curve denotes a meridian.

and we set $\mathcal{M}$ as the complement of $\mathcal{K}$. The Pretzel knot (−2,3,7) has the same hyperbolic volume with 5_2, $\mathrm{Vol}(\mathcal{M}) = 2.82812\ldots$, but the triangulations of the complement give the following partition function:

$$Z_\gamma\left(\mathcal{M}_u\right) = \int_{\mathbb{R}} \mathrm{d}p\,\delta_C(p;u)\langle p_4, p_7|S|p_6, p_1\rangle\langle p_5, p_8|S|p_7, p_5\rangle\langle p_1, p_6|S|p_8, p_2\rangle\langle p_3, p_2|S|p_4, p_3\rangle. \tag{4.15}$$

Note that this triangulation differs from the canonical triangulation in Ref. [60]. The developing map is given in Fig. 5. Here we set the moduli of four ideal tetrahedra as

$$z_1 = \mathrm{e}^{p_1 - p_7} \qquad z_2 = \mathrm{e}^{p_5 - p_8}$$
$$z_3 = \mathrm{e}^{p_2 - p_6} \qquad z_4 = \mathrm{e}^{p_3 - p_2}.$$

Then the meridian is read as

$$p_2 - p_6 - p_1 + p_7 = -2u.$$

In the classical limit of the partition function $Z_\gamma(\mathcal{M}_u)$, we obtain the potential function after some changes of variables as

$$V_{\mathcal{M}}(x,y,z;m) = -\frac{2\pi^2}{3} + Li_2\left(\frac{1}{z}\right) + Li_2\left(\frac{1}{xyzm^4}\right) + Li_2\left(\frac{1}{zm^2}\right) + Li_2\left(\frac{m^2}{x}\right)$$
$$+ 3\left(\log(m^2)\right)^2 + \log\left(\frac{y^5}{x^2}\right)\log\left(m^2\right) + (\log x)^2 + (\log y)^2. \tag{4.16}$$

The hyperbolic consistency conditions (3.10) give

$$\begin{aligned} &\left(xyzm^4 - 1\right)\left(x - m^2\right) = yzm^8 \\ &ym^6\left(xyzm^4 - 1\right) = xz \\ &(z-1)\left(xyzm^4 - 1\right)\left(m^2 z - 1\right) = xyz^3 m^6 \end{aligned} \tag{4.17}$$

and the longitude (3.11) is defined by

$$\ell = \frac{\left(xyzm^4 - 1\right)^2\left(m^2 z - 1\right)^2 m^2 y^3}{\left(x - m^2\right) x^3 z^3}. \tag{4.18}$$

In the complete hyperbolic structure $m = 1$, we have a solution of (4.17),

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.337641 - 0.56228\mathrm{i} \\ 0.122561 + 0.744862\mathrm{i} \\ 0.618504 - 0.410401\mathrm{i} \end{pmatrix}$$

s.t. we recover the hyperbolic volume of $\mathcal{K}$:

$$\operatorname{Im} V_{\mathcal{M}}(x, y, z; m = 1) = 2D(1/z) + D\left(\frac{1}{xyz}\right) + D(1/x) = 2.82812\ldots.$$

By replacing the Bloch–Wigner function by the Rogers dilogarithm function and choosing a branch such that the imaginary part coincides with the volume, we recover the Chern–Simons term as

$$\begin{aligned} V_{\mathcal{M}}(x, y, z; m = 1) &= -\frac{2\pi^2}{3} + 2L(1/z) + L\left(\frac{1}{xyz}\right) + L(1/x) + \pi\mathrm{i}\log(xz) \\ &= 2\pi^2 \cdot 0.236537\ldots + \mathrm{i} \cdot 2.82812\ldots. \end{aligned}$$

The A-polynomial is computed by eliminating (x, y, z) from (4.17) and (4.18), and we obtain $A_{\mathcal{M}}(\ell, m) = 0$, where the A-polynomial for the $(-2, 3, 7)$ Pretzel knot is

$$\begin{aligned} A_{\mathcal{M}}(\ell, m) = &-1 + \left(m^{16} - 2m^{18} + m^{20}\right)\ell + \left(2m^{36} + m^{38}\right)\ell^2 \\ &- \ell^4\left(m^{72} + 2m^{74}\right) - \ell^5\left(m^{90} - 2m^{92} + m^{94}\right) + m^{110}\ell^6. \end{aligned} \tag{4.19}$$

4.4. Once-punctured torus bundles over the circle

One of benefits of the partition function $Z_\gamma(\mathcal{M})$ is that we can compute it explicitly, at least the asymptotics thereof, for a once-punctured torus bundle over S^1. A once-punctured torus bundle over S^1, which we denote $\mathcal{M}(\varphi)$, is described by $F \times [0, 1]/(x, 0) \sim (\varphi(x), 1)$ where a monodromy matrix $\varphi \in SL(2, \mathbb{Z})$ is a homeomorphism from the punctured torus $F = \mathbb{T}^2 \setminus \{0\}$ to itself. Thurston's hyperbolization theorem indicates that $\mathcal{M}(\varphi)$ admits a complete hyperbolic metric with a finite volume when φ has 2 distinct real eigenvalues. In this case, the monodromy matrix φ can be written up to conjugation as

$$\varphi = L^{s_1} R^{t_1} \cdots L^{s_n} R^{t_n} \tag{4.20}$$

where

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

with $n > 0$, and s_j and t_j are positive integers. Note that the complement of the figure-eight knot studied in Section 4.1 corresponds to $\varphi = LR = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

It is known that we can triangulate the manifold $M(\varphi)$ with $\sum_{k=1}^{n}(s_k + t_k)$ ideal tetrahedra [19], and we have the partition function as

$$\begin{aligned} Z_\gamma\left(\mathcal{M}_u(\varphi)\right) = &\iiiint_{\mathbb{R}} \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c\,\mathrm{d}d\;\delta_C(a, b, c, d; u)\delta_G(a, b, c, d) \\ &\times \prod_{k=1}^{n}\left[\prod_{i=0}^{s_k-1}\langle d_{k,i}, c_{k,i+1}|S^{-1}|d_{k,i+1}, c_{k,i}\rangle \prod_{j=0}^{t_k-1}\langle b_{k,j}, a_{k,j+1}|S|b_{k,j+1}, a_{k,j}\rangle\right] \end{aligned} \tag{4.21}$$

where a gluing condition $\delta_G(a, b, c, d)$ means

$$\begin{cases} b_{k,0} = c_{k+1,0} \\ a_{k,0} = d_{k+1,0} \end{cases} \quad \text{for } k = 1, 2, \ldots, n-1 \qquad \begin{cases} b_{n,0} = c_{1,0} \\ a_{n,0} = d_{1,0} \end{cases}$$

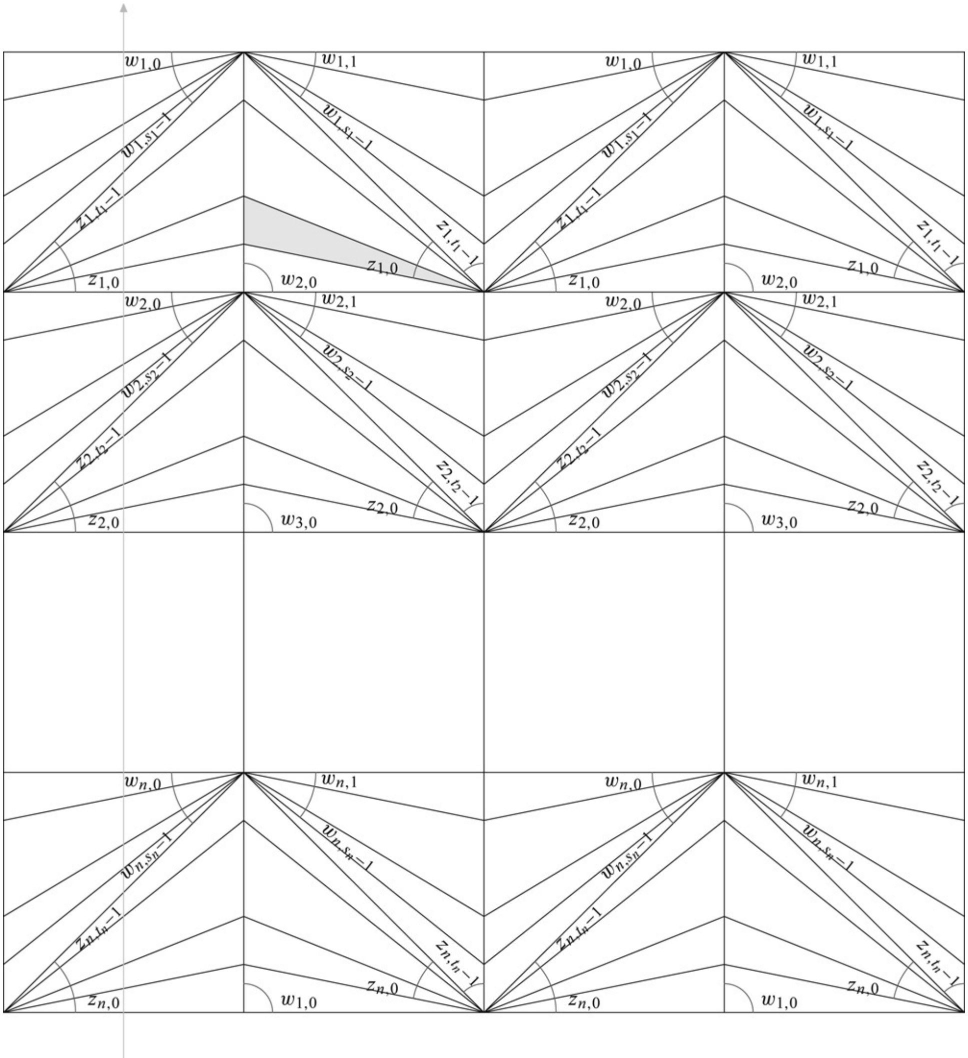


Fig. 6. Schematic developing map for $\mathcal{M}(\varphi)$. Top vertex of tetrahedron with modulus $z_{1,0}$ is filled gray. Gray straight line denotes meridian.

$$\begin{cases} b_{k,t_k} = c_{k,s_k} \\ a_{k,t_k} = d_{k,s_k} \end{cases} \quad \text{for } k = 1, 2, \ldots, n.$$

When we set the modulus of each oriented tetrahedron as

$$z_{k,j} = \mathrm{e}^{a_{k,j} - a_{k,j+1}} \qquad w_{k,j} = \mathrm{e}^{c_{k,j} - c_{k,j+1}}$$

the developing map is drawn schematically as Fig. 6. We can then read a condition $\delta_C(a, b, c, d; u)$ for meridian as

$$u = \frac{1}{2} \sum_{k=1}^{n} \left(c_{k,0} - c_{k,s_k} - a_{k,0} + a_{k,t_k} \right). \tag{4.22}$$

With these conditions, we have the partition function $Z_\gamma\ (\mathcal{M}_u(\varphi))$, and the Neumann–Zagier potential function can be given by taking the classical limit $\gamma \to 0$. As far as we know, both the quantum invariant and the A-polynomial-type invariant for the once-punctured torus bundle over S^1 have not been studied, but we can obtain the A-polynomial from the Neumann–Zagier function as a classical limit of the partition function.

Below we give a few examples for concreteness.

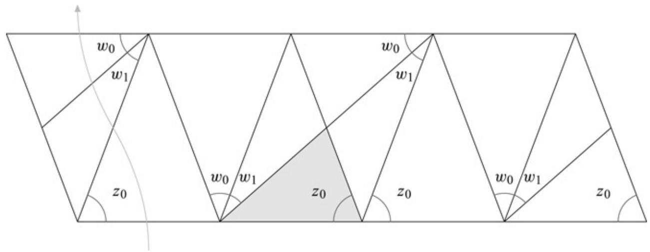


Fig. 7. Developing map of $\mathcal{M}(L^2R)$. Gray filled triangle is top vertex of the tetrahedron with modulus z_0. Gray curve is the meridian.

4.4.1. L^2R

We set $\varphi = L^2R = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$, whose hyperbolic Dehn surgery is studied in Ref. [6]. The partition function (4.21) is rewritten as

$$Z_\gamma\left(\mathcal{M}_u(L^2R)\right) = \int_{\mathbb{R}} \mathrm{d}\, p \delta_C(p; u) \langle p_1, p_5 | S^{-1} | p_6, p_3\rangle \langle p_6, p_4 | S^{-1} | p_2, p_5\rangle \langle p_3, p_2 | S | p_4, p_1\rangle. \tag{4.23}$$

When we set the modulus of tetrahedra as $w_0 = \mathrm{e}^{p_3 - p_5}$, $w_1 = \mathrm{e}^{p_5 - p_4}$, and $z_0 = \mathrm{e}^{p_1 - p_2}$, the developing map $\mathcal{M}(L^2R)$ can be depicted as Fig. 7. Then the meridian is read as

$$p_3 - p_4 - p_1 + p_2 = 2u. \tag{4.24}$$

In the classical limit, we have

$$Z_\gamma\left(\mathcal{M}_u(L^2R)\right) \sim \iint \mathrm{d}x\mathrm{d}y \exp\left(\frac{1}{2\mathrm{i}\gamma} V_{\mathcal{M}(L^2R)}(\mathrm{e}^x, \mathrm{e}^y; \mathrm{e}^u)\right) \tag{4.25}$$

where the potential function is computed as

$$\begin{aligned} V_{\mathcal{M}(L^2R)}(x, y; m) = {} & Li_2\left(\frac{1}{m^2 x}\right) - Li_2\left(\frac{1}{m^2 x y^2}\right) - Li_2\left(m^2 x^2 y^2\right) \\ & - \log\left(m^2\right) \log\left(m^2 x^2 y^4\right) - 2\left(\log(xy)\right)^2 + \frac{\pi^2}{6}. \end{aligned} \tag{4.26}$$

The saddle point conditions (3.10), which correspond to the hyperbolic consistency condition, reduce to

$$\begin{aligned} & \frac{\left(-1 + m^2 x\right)\left(-1 + m^2 x^2 y^2\right)^2}{m^4 x^4 y^2 \left(-1 + m^2 x y^2\right)} = 1 \\ & \frac{\left(-1 + m^2 x^2 y^2\right)}{m^2 x \left(-1 + m^2 x y^2\right)} = 1 \end{aligned} \tag{4.27}$$

and, as (3.11), the longitude is defined by

$$\ell = -\frac{(-1 + m^2 x)(-1 + m^2 x^2 y^2)}{m^4 x^2 y^2 (-1 + m^2 x y^2)}. \tag{4.28}$$

In the complete case $m = 1$, we find that $(x, y^2) = \left(\frac{1 \pm \sqrt{7}\mathrm{i}}{4}, \frac{-1 \pm \sqrt{7}\mathrm{i}}{2}\right)$ solves the consistency condition (4.27). Among these, we can check numerically that the largest value of the imaginary part of the potential function at the saddle points

$$\operatorname{Im} V_{\mathcal{M}(L^2R)}(x, y; m = 1) = D\left(\frac{1}{x}\right) - D\left(\frac{1}{x y^2}\right) - D\left(x^2 y^2\right)$$

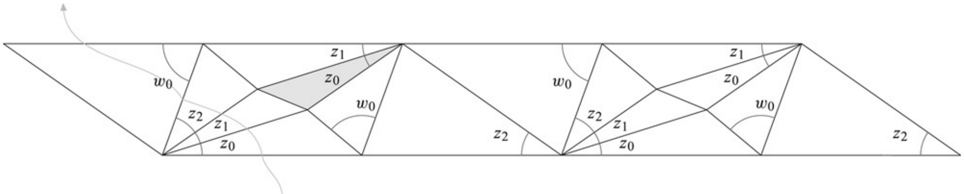


Fig. 8. Developing map of $\mathcal{M}(LR^3)$.

coincides with the hyperbolic volume $\mathrm{Vol}\left(\mathcal{M}(L^2R)\right) = 2.66674\ldots$, and we have

$$\begin{aligned} V_{\mathcal{M}(L^2R)}(x, y; m = 1) &= \frac{\pi^2}{6} + L\left(\frac{1}{x}\right) - L\left(\frac{1}{xy^2}\right) - L\left(x^2y^2\right) \\ &= 2\pi^2 \cdot 0.0208333\ldots + \mathrm{i} \cdot 2.66674\ldots. \end{aligned}$$

The A-polynomial is now given by eliminating x and y from the set of equations, (4.27) and (4.28), and we obtain the algebraic curve $A_{\mathcal{M}(L^2R)}(\ell, m) = 0$ whose Newton polygon is given by:

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 2 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

4.4.2. LR^3

We take another example, $\varphi = LR^3 = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$. In this case, the partition function (4.21) becomes

$$\begin{aligned} Z_\gamma(\mathcal{M}_u(LR^3)) = \int_{\mathbb{R}} \mathrm{d}p \delta_C(p; u) &\langle p_1, p_4 | S | p_3, p_2 \rangle \langle p_3, p_6 | S | p_5, p_4 \rangle \\ &\times \langle p_5, p_8 | S | p_7, p_6 \rangle \langle p_2, p_7 | S^{-1} | p_8, p_1 \rangle. \end{aligned} \tag{4.29}$$

We set the modulus of each tetrahedron as

$$\begin{aligned} z_0 &= \mathrm{e}^{p_2 - p_4} & z_1 &= \mathrm{e}^{p_4 - p_6} \\ z_2 &= \mathrm{e}^{p_6 - p_8} & w_0 &= \mathrm{e}^{p_1 - p_7}. \end{aligned}$$

The developing map is depicted in Fig. 8, and the meridian is read as $\frac{w_0}{z_0 z_1 z_2}$. We thus have a condition of $\delta_C(p; u)$ as

$$p_1 - p_7 - p_2 + p_8 = 2u. \tag{4.30}$$

In the classical limit, we obtain

$$Z_\gamma(\mathcal{M}_u(LR^3)) \sim \iiint \mathrm{d}x\mathrm{d}y\mathrm{d}z \exp\left(\frac{1}{2\mathrm{i}\gamma} V_{\mathcal{M}(LR^3)}(\mathrm{e}^x, \mathrm{e}^y, \mathrm{e}^z; \mathrm{e}^u)\right) \tag{4.31}$$

where the potential function is computed as

$$\begin{aligned} V_{\mathcal{M}(LR^3)}(x, y, z; m) = &- Li_2\left(\frac{1}{m^4 y}\right) + Li_2\left(\frac{1}{m^2 x y z^2}\right) + Li_2\left(m^2 x^2 z\right) \\ &+ Li_2\left(\frac{m^2 y^2 z}{x}\right) - \left(\log\left(m^2\right)\right)^2 + 2\log\left(\frac{x}{m^2}\right)\log\left(\frac{x}{y}\right) \\ &+ 2\log y \log(yz) + 2\log z \log(xzm^2) - \frac{\pi^2}{3}. \end{aligned} \tag{4.32}$$

In the complete case of $m = 1$, we can check numerically that among algebraic solutions of (3.10), the maximum of the imaginary part of $V_{\mathcal{M}(LR^3)}(x, y, z; m = 1)$,

$$\operatorname{Im} V_{\mathcal{M}(LR^3)}(x, y, z; 1) = -D\left(\frac{1}{y}\right) + D\left(\frac{1}{xyz^2}\right) + D\left(x^2 z\right) + D\left(\frac{y^2 z}{x}\right)$$

coincides with the hyperbolic volume $\mathrm{Vol}(\mathcal{M}(LR^3)) = 2.98912\ldots$, and we have

$$\begin{aligned} V_{\mathcal{M}(LR^3)}(x, y, z; 1) &= -L\left(\frac{1}{y}\right) + L\left(\frac{1}{xyz^2}\right) + L\left(x^2 z\right) + L\left(\frac{y^2 z}{x}\right) - \frac{\pi^2}{3} \\ &= -2\pi^2 \cdot 0.0368931\ldots + \mathrm{i} \cdot 2.98912\ldots \end{aligned}$$

where

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.475468 + 0.621671\mathrm{i} \\ 0.203723 + 0.560668\mathrm{i} \\ 0.572495 - 1.57557\mathrm{i} \end{pmatrix}.$$

See Ref. [20], where algebraic solutions of the consistency equations are investigated in detail.

Correspondingly we obtain the A-polynomial for $\mathcal{M}(LR^3)$ by eliminating x, y, and z from a set of equations. Explicitly the polynomial $A_{\mathcal{M}(LR^3)}(\ell, m)$ is given in the form of Newton polygon as follows:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ -1 & -2 & -3 & -1 & -2 & 0 \\ 0 & -3 & -2 & 2 & 3 & 0 \\ 0 & 2 & 1 & 3 & 2 & 1 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

5. Conclusion and discussion

We have constructed a partition function $Z_\gamma(\mathcal{M})$ for cusped hyperbolic 3-manifolds $\mathcal{M}$ by assigning the Faddeev quantum dilogarithm function to oriented ideal tetrahedra. Once the triangulation of the cusped 3-manifold $\mathcal{M}$ is given, it is rather straightforward to define the partition function $Z_\gamma(\mathcal{M})$ in an integral form. In the classical limit, the Faddeev integral reduces to the dilogarithm function, whose imaginary part denotes the hyperbolic volume of the ideal tetrahedron. What is remarkable is that the saddle point conditions coincide with the hyperbolic consistency conditions around edges [24]. We have discussed, as a variant of the volume conjecture, that the partition function $Z_\gamma(\mathcal{M})$, which can be regarded as a generalization of the Kashaev invariant (specific value of the colored Jones polynomial) and quantum hyperbolic invariant [3], is dominated by the hyperbolic volume in the classical limit $\gamma \to 0$.

We have shown that the partition function can be defined even for a one-parameter deformation of manifold $\mathcal{M}_u$ (not complete), and that the Neumann–Zagier potential function can be given by taking a classical limit $\gamma \to 0$. Correspondingly, the A-polynomial can be computed from the potential function (3.13). This may support the generalized volume conjecture (1.2) proposed in Ref. [22]. We have demonstrated by giving examples that our method recovers a previously known A-polynomial when $\mathcal{M}$ is a complement of hyperbolic knots. We have further applied our method for the once-punctured torus bundle over the circle. It seems that the A-polynomial-type invariant has not been known for this 3-manifold, and it will be interesting to study a relationship with the boundary slope. To conclude, our results indicate that the S-operator (2.13) denotes the quantum Bloch invariant for an oriented ideal tetrahedron.

The A-polynomial may provide an interesting insight for mathematical physics. For example, the Mahler measure of the A-polynomial is expected to coincide with the hyperbolic volume of the knot. On the other hand, the Mahler measure of the determinant of the Kasteleyn matrix has appeared as the free energy of the dimer problem. From the viewpoint of our combinatorial construction, the partition function based on oriented ideal triangulation may be interpreted as a matching of in-states and out-states, and it might be interesting to investigate a Kasteleyn matrix interpretation of the A-polynomial.

Acknowledgments

The author would like to thank H. Murakami, K. Shimokawa, and T. Takata for communications. He also thanks R. Benedetti for bringing Refs. [2,3] to attention. We have used the computer programs SnapPea [60], Knotscape [28], and Snap [14], in studying triangulation of manifolds. We have also used `Mathematica` and Pari/GP. Pictures of knots in this paper are drawn using KnotPlot [54]. This work is supported in part by a Grant-in-Aid from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

Appendix. More examples

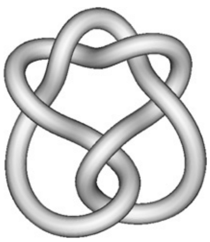
We shall study the partition function for other cusped manifolds in the following. We give a list of

- the partition function $Z_\gamma(\mathcal{M}_u)$ in terms of the S-operators,
- the developing map when the number of the ideal tetrahedra is less than four,
- the condition $\delta_C(p; u)$, where u is a deformation parameter from the completeness $u = 0$,
- the Neumann–Zagier potential function $V_\mathcal{M}(x; m)$, which follows from $Z_\gamma(\mathcal{M}_u)$ by taking the classical limit $\gamma \to 0$,
- a solution of the saddle point equations in the complete case $u = 0$, in which it is checked that the hyperbolic volume coincides with the maximal value of the imaginary part of the potential function at this saddle point. We further replace the Bloch–Wigner function with the Rogers dilogarithm function and choose a branch so that the imaginary (resp. real) part gives the hyperbolic volume (resp. the Chern–Simons invariant modulo π^2),
- the A-polynomial, which is given from the potential function $V_\mathcal{M}(x; m)$ by eliminating parameters x.

In the first part Appendix A.1, we collect results for hyperbolic knots up to 7 crossings. Though the A-polynomial is given in Ref. [12], we give an explicit form for self-completeness. In the second part Appendix A.2, we consider *simple* hyperbolic knots Kx_y from Ref. [9]. Pictures of knots may be complicated, but the triangulation of their complements is relatively simple in these cases. It should be remarked that the A-polynomial and the Chern–Simons invariant ($\mathrm{CS} = 2\pi^2 \cdot \mathrm{cs}$ and cs is defined modulo 1/2) may differ from the results in Refs. [12,9] due to opposite orientation. We note that though the number of the ideal tetrahedra of the complement of knot Kx_y is x in the canonical triangulation [60], our triangulations are different.

A.1. Complement of knots up to 7 crossings

A.1.1. 6_1



$$\begin{cases} \mathrm{Vol}(S^3 \setminus \mathcal{K}) = 3.16396\ldots \\ \mathrm{cs}(S^3 \setminus \mathcal{K}) = 0.1559770167\ldots \bmod 1/2. \end{cases}$$

- Partition function

$$Z_\gamma(\mathcal{M}_u) = \int_\mathbb{R} \mathrm{d}p\,\delta_C(p; u)\langle p_3, p_1|S^{-1}|p_2, p_4\rangle\langle p_6, p_4|S|p_5, p_8\rangle\langle p_7, p_5|S^{-1}|p_6, p_1\rangle\langle p_8, p_2|S|p_7, p_3\rangle.$$

- Developing map (Fig. 9).
- Condition $\delta_C(p; u)$

$$p_4 - p_1 - p_3 + p_2 = 2u.$$

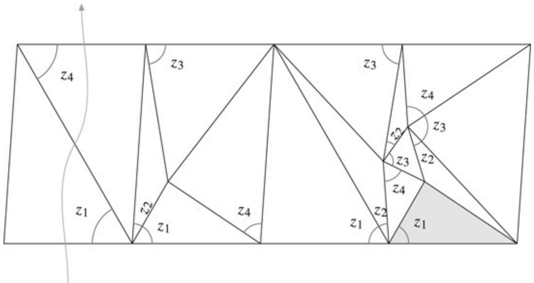


Fig. 9. Developing map of the knot complement of 6_1. Gray triangle is a horosphere of the top vertex of the tetrahedron with modulus z_1, where we have set $z_1 = \mathrm{e}^{p_4 - p_1}$, $z_2 = \mathrm{e}^{p_8 - p_4}$, $z_3 = \mathrm{e}^{p_1 - p_5}$, and $z_4 = \mathrm{e}^{p_3 - p_2}$.

- Potential function

$$V_{\mathcal{M}}(x, y, z; m) = Li_2\left(\frac{zm^2}{xy}\right) + Li_2(y) - Li_2\left(\frac{m^2}{y}\right) - Li_2\left(\frac{yz}{m^2}\right) + \log\left(m^2\right)\log\left(\frac{yz^2}{x}\right) - \log z \log(xy).$$

- Hyperbolic volume & Chern–Simons

$$\operatorname{Im} V_{\mathcal{M}}(x, y, z; m = 1) = D\left(\frac{z}{xy}\right) + D(y) - D\left(\frac{1}{y}\right) - D(yz)$$

$$\begin{aligned} V_{\mathcal{M}}(x, y, z; m = 1) &= L\left(\frac{z}{xy}\right) + L(y) - L\left(\frac{1}{y}\right) - L(yz) - \pi \mathrm{i} \log(z) \\ &= -2\pi^2 \cdot 0.344023\ldots + \mathrm{i} \cdot 3.16396\ldots \end{aligned}$$

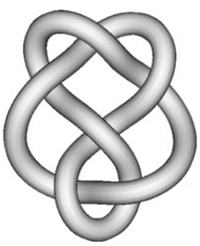
where

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -0.851808 + 0.911292\mathrm{i} \\ 0.278726 + 0.48342\mathrm{i} \\ -1.50411 - 1.22685\mathrm{i} \end{pmatrix}.$$

- A-polynomial

$$\begin{pmatrix} 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & -3 & -3 & 0 & 0 \\ -1 & -3 & -6 & -3 & -1 \\ 0 & 0 & -3 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & -1 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix}.$$

A.1.2. 6_2



$$\begin{cases} \mathrm{Vol}(S^3 \setminus \mathcal{K}) = 4.40083\ldots \\ \mathrm{cs}(S^3 \setminus \mathcal{K}) = 0.2024924984\ldots \bmod 1/2. \end{cases}$$

- Partition function

$$Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p\,\delta_C(p;u)\langle p_8, p_6|S^{-1}|p_1, p_2\rangle\langle p_3, p_7|S^{-1}|p_8, p_4\rangle \times \langle p_4, p_5|S|p_3, p_9\rangle\langle p_2, p_{10}|S|p_6, p_5\rangle\langle p_1, p_9|S^{-1}|p_{10}, p_7\rangle.$$

- Condition $\delta_C(p;u)$

$$p_2 - p_6 - p_5 + p_{10} = -2u.$$

- Potential function

$$V_{\mathcal{M}}(w,x,y,z;m) = \frac{\pi^2}{6} + Li_2\left(\frac{z}{m^2}\right) + Li_2\left(\frac{m^4}{yw}\right) - Li_2\left(\frac{yw}{m^2}\right) - Li_2\left(\frac{m^2}{xy}\right) - Li_2\left(\frac{yz}{m^2}\right) + \log x \log z - \log(yw)\log(yz) + \log\left(m^2\right)\log\left(\frac{y^2 z}{w}\right).$$

- Hyperbolic volume

$$\operatorname{Im} V_{\mathcal{M}}(w,x,y,z;1) = D(z) + D\left(\frac{1}{yw}\right) - D(yw) - D\left(\frac{1}{xy}\right) - D(yz)$$

$$V_{\mathcal{M}}(w,x,y,z;1) = \frac{\pi^2}{6} + L(z) + L\left(\frac{1}{yw}\right) - L(yw) - L\left(\frac{1}{xy}\right) - L(yz) = 2\pi^2 \cdot 0.297508\ldots + \mathrm{i}\cdot 4.40083\ldots$$

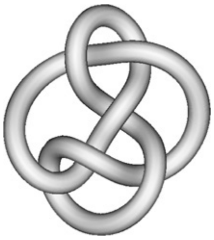
with

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -0.455697 + 1.20015\mathrm{i} \\ -0.964913 - 0.621896\mathrm{i} \\ -0.418784 - 0.219165\mathrm{i} \\ 0.0904327 + 1.60288\mathrm{i} \end{pmatrix}.$$

- A-polynomial

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & -2 & 1 & 0 & 0 & 0 \\
-1 & 1 & -3 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 \\
0 & -5 & 5 & 0 & 0 & 0 \\
0 & -5 & 3 & -3 & 0 & 0 \\
0 & 3 & -12 & 8 & 0 & 0 \\
0 & 0 & -13 & 3 & 0 & 0 \\
0 & 0 & 3 & -13 & 0 & 0 \\
0 & 0 & 8 & -12 & 3 & 0 \\
0 & 0 & -3 & 3 & -5 & 0 \\
0 & 0 & 0 & 5 & -5 & 0 \\
0 & 0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & -3 & 1 & -1 \\
0 & 0 & 0 & 1 & -2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.$$

A.1.3. 6_3



$$\begin{cases}\mathrm{Vol}(S^3 \setminus \mathcal{K}) = 5.69302\ldots \\ \mathrm{cs}(S^3 \setminus \mathcal{K}) = 0.0 \bmod 1/2.\end{cases}$$

- Partition function

$$Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p\, \delta_C(p; u) \langle p_3, p_2 | S | p_1, p_9 \rangle \langle p_7, p_9 | S^{-1} | p_5, p_{10} \rangle \langle p_{10}, p_4 | S^{-1} | p_6, p_8 \rangle \\ \times \langle p_8, p_1 | S^{-1} | p_4, p_{11} \rangle \langle p_{12}, p_{11} | S^{-1} | p_2, p_3 \rangle \langle p_6, p_5 | S | p_{12}, p_7 \rangle. \tag{A.1}$$

- Condition $\delta_C(p; u)$

$$p_{10} - p_9 - p_7 + p_5 = -2u.$$

- Potential function

$$V_{\mathcal{M}}(v, w, x, y, z; m) = Li_2\left(\frac{m^2 y}{x}\right) + Li_2\left(\frac{v}{w}\right) - Li_2\left(\frac{m^2 w}{v}\right) - Li_2\left(\frac{xz}{v}\right) - Li_2\left(\frac{x^2 z}{v}\right) - Li_2\left(\frac{v}{xyz}\right) \\ + \frac{\pi^2}{3} + \log\left(m^2\right)\log\left(\frac{y}{w}\right) + \log x \log\left(\frac{v}{yz^2}\right) + \log\left(\frac{w}{yz}\right)\log\left(\frac{z}{v}\right).$$

- Hyperbolic volume

$$\operatorname{Im} V_{\mathcal{M}}(v, w, x, y, z; 1) = D\left(\frac{y}{x}\right) + D\left(\frac{v}{w}\right) - D\left(\frac{w}{v}\right) - D\left(\frac{xz}{v}\right) - D\left(\frac{x^2 z}{v}\right) - D\left(\frac{v}{xyz}\right)$$

$$V_{\mathcal{M}}(v, w, x, y, z; 1) = \frac{\pi^2}{3} + L\left(\frac{y}{x}\right) + L\left(\frac{v}{w}\right) - L\left(\frac{w}{v}\right) - L\left(\frac{xz}{v}\right) - L\left(\frac{x^2 z}{v}\right) - L\left(\frac{v}{xyz}\right) \\ = 0.0 + \mathrm{i} \cdot 5.69302\ldots$$

with

$$\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.0739495 + 0.558752\mathrm{i} \\ 0.732786 + 0.381252\mathrm{i} \\ 1.0 \\ 0.108378 + 0.818891\mathrm{i} \\ 0.415113 + 0.381252\mathrm{i} \end{pmatrix}.$$

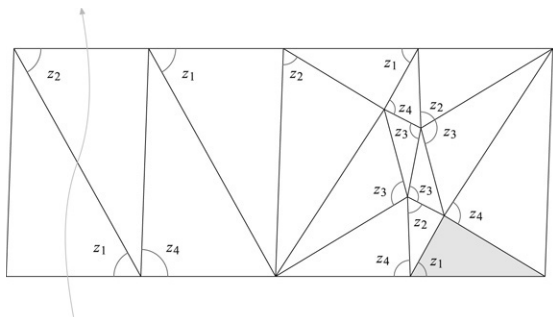
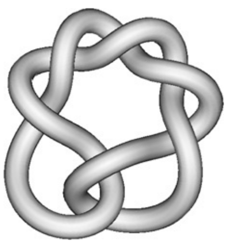


Fig. 10. Developing map of the knot complement of 7_2. The gray triangle corresponds to top vertex of the tetrahedron with modulus z_1, where we have set modulus as $z_1 = \mathrm{e}^{p_3 - p_4}$, $z_2 = \mathrm{e}^{p_1 - p_2}$, $z_3 = \mathrm{e}^{p_6 - p_7}$, and $z_4 = \mathrm{e}^{p_5 - p_3}$.

- A-polynomial

$$\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -5 & 1 & 0 & 0 \\
0 & 0 & -4 & 3 & -4 & 0 & 0 \\
0 & 0 & 4 & 9 & 4 & 0 & 0 \\
0 & 2 & 2 & -2 & 2 & 2 & 0 \\
0 & -5 & -6 & -21 & -6 & -5 & 0 \\
0 & 1 & 2 & 8 & 2 & 1 & 0 \\
1 & 10 & 17 & 34 & 17 & 10 & 1 \\
0 & 1 & 2 & 8 & 2 & 1 & 0 \\
0 & -5 & -6 & -21 & -6 & -5 & 0 \\
0 & 2 & 2 & -2 & 2 & 2 & 0 \\
0 & 0 & 4 & 9 & 4 & 0 & 0 \\
0 & 0 & -4 & 3 & -4 & 0 & 0 \\
0 & 0 & 1 & -5 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.$$

A.1.4. 7_2



$$\begin{cases} \mathrm{Vol}(S^3 \setminus \mathcal{K}) = 3.33174\ldots \\ \mathrm{cs}(S^3 \setminus \mathcal{K}) = 0.0551535349\ldots \bmod 1/2. \end{cases}$$

- Partition function

$$Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p\,\delta_C(p;u) \langle p_1, p_4 | S | p_2, p_3 \rangle \langle p_5, p_2 | S | p_8, p_1 \rangle \langle p_8, p_7 | S^{-1} | p_7, p_6 \rangle \langle p_6, p_3 | S | p_4, p_5 \rangle.$$

- Developing map (Fig. 10).
- Condition $\delta_C(p;u)$

$$p_3 - p_4 - p_1 + p_2 = -2u.$$

- Potential function

$$V_{\mathcal{M}}(x, y, z; m) = Li_2\left(\frac{1}{x}\right) + Li_2\left(\frac{m^2}{x}\right) + Li_2\left(zm^2\right) - Li_2\left(\frac{yz}{x^2}\right) - \frac{\pi^2}{3}$$

$$- (\log(x/y))^2 + \log z \log\left(\frac{x^3 m^2}{y^2 z}\right).$$

- Hyperbolic volume

$$\operatorname{Im} V_{\mathcal{M}}(x, y, z; 1) = 2D\left(\frac{1}{x}\right) + D(z) - D\left(\frac{yz}{x^2}\right)$$

$$\begin{aligned} V_{\mathcal{M}}(x, y, z; 1) &= 2L\left(\frac{1}{x}\right) + L(z) - L\left(\frac{yz}{x^2}\right) - \frac{\pi^2}{3} + \pi \mathrm{i} \log\left(\frac{x}{yz}\right) \\ &= -2\pi^2 \cdot 0.555154\ldots + \mathrm{i} \cdot 3.33174\ldots \end{aligned}$$

with

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.941819 - 1.69128\mathrm{i} \\ 0.935538 + 0.903908\mathrm{i} \\ 0.0581814 + 1.69128\mathrm{i} \end{pmatrix}.$$

- A-polynomial

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 4 & -4 & 0 & 0 & 0 \\ 0 & 3 & 2 & -2 & 0 & 0 \\ 0 & 0 & 5 & 5 & 0 & 0 \\ 0 & 0 & 6 & 1 & 1 & 0 \\ 0 & 0 & 0 & -4 & -1 & 0 \\ 0 & -1 & -4 & 0 & 0 & 0 \\ 0 & 1 & 1 & 6 & 0 & 0 \\ 0 & 0 & 5 & 5 & 0 & 0 \\ 0 & 0 & -2 & 2 & 3 & 0 \\ 0 & 0 & 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{pmatrix}.$$

A.1.5. 7_3

$$\begin{cases} \operatorname{Vol}(S^3 \setminus \mathcal{K}) = 4.5921256970\ldots \\ \operatorname{cs}(S^3 \setminus \mathcal{K}) = 0.1872201781\ldots \bmod 1/2. \end{cases}$$

- Partition function

$$\begin{aligned} Z_\gamma(\mathcal{M}_u) = &\int_{\mathbb{R}} \mathrm{d}p\, \delta_C(p; u) \langle p_1, p_3|S|p_4, p_8\rangle \langle p_2, p_8|S|p_5, p_1\rangle \\ &\times \langle p_5, p_9|S^{-1}|p_6, p_7\rangle \langle p_4, p_6|S|p_3, p_{10}\rangle \langle p_7, p_{10}|S|p_2, p_9\rangle. \end{aligned}$$

- Condition $\delta_C(p; u)$

$$p_{10} - p_6 - p_1 + p_8 = -2u.$$

- Potential function

$$\begin{aligned} V_{\mathcal{M}}(w, x, y, z; m) = &\, Li_2\left(m^2 w\right) + Li_2(m^4 w) + Li_2\left(\frac{1}{m^2 x}\right) + Li_2\left(\frac{z}{w}\right) - Li_2\left(\frac{z}{y}\right) - \frac{\pi^2}{2} \\ &+ \log w \log x + \log y \log z + \log\left(m^2\right) \log(y w^2 z). \end{aligned}$$

- Hyperbolic volume

$$\operatorname{Im} V_{\mathcal{M}}(w, x, y, z; 1) = 2D(w) + D\left(\frac{1}{x}\right) + D\left(\frac{z}{w}\right) - D\left(\frac{z}{y}\right)$$

$$\begin{aligned} V_{\mathcal{M}}(w, x, y, z; 1) &= -\frac{\pi^2}{2} + 2L(w) + L\left(\frac{1}{x}\right) + L\left(\frac{z}{w}\right) - L\left(\frac{z}{y}\right) - \pi \mathrm{i} \log z \\ &= 2\pi^2 \cdot 0.18722\ldots + \mathrm{i} \cdot 4.59213\ldots \end{aligned}$$

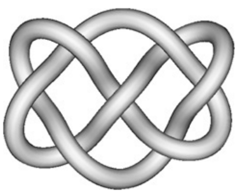
with

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0.645284 - 0.801205\mathrm{i} \\ -0.676708 + 0.260961\mathrm{i} \\ -0.87287 + 1.51178\mathrm{i} \\ 0.537981 + 1.04357\mathrm{i} \end{pmatrix}.$$

- A-polynomial

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -5 & -3 & 0 & 0 & 0 & 0 \\ 0 & -2 & 9 & 0 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & -14 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & -10 & 0 & 0 & 0 \\ 0 & 0 & -4 & 3 & 0 & 0 & 0 \\ 0 & 0 & -2 & 12 & 0 & 0 & 0 \\ 0 & 0 & -3 & -6 & -1 & 0 & 0 \\ 0 & 0 & 3 & 24 & 3 & 0 & 0 \\ 0 & 0 & -1 & -6 & -3 & 0 & 0 \\ 0 & 0 & 0 & 12 & -2 & 0 & 0 \\ 0 & 0 & 0 & 3 & -4 & 0 & 0 \\ 0 & 0 & 0 & -10 & 4 & 0 & 0 \\ 0 & 0 & 0 & 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -14 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 9 & -2 & 0 \\ 0 & 0 & 0 & 0 & -3 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

A.1.6. 7_4



$$\begin{cases} \mathrm{Vol}(S^3 \setminus \mathcal{K}) = 5.13794\ldots \\ \mathrm{cs}(S^3 \setminus \mathcal{K}) = 0.02172669\ldots \bmod 1/2. \end{cases}$$

- Partition function

$$Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p\,\delta_C(p;u)\langle p_1, p_{10}|S|p_3, p_4\rangle\langle p_9, p_2|S|p_{12}, p_1\rangle\langle p_4, p_{11}|S^{-1}|p_5, p_6\rangle \times \langle p_3, p_5|S^{-1}|p_2, p_7\rangle\langle p_{12}, p_8|S|p_{10}, p_9\rangle\langle p_6, p_7|S|p_8, p_{11}\rangle.$$

- Condition $\delta_C(p;u)$

$$p_4 - p_{10} + p_7 - p_5 - p_1 + p_2 = -2u.$$

- Potential function

$$\begin{aligned} V_{\mathcal{M}}(v,w,x,y,z;m) = {} & Li_2\left(\frac{w}{x}\right) + Li_2\left(\frac{vw}{y}\right) - Li_2\left(\frac{vxm^2}{y}\right) \\ & + Li_2\left(\frac{m^4x}{wy}\right) - Li_2\left(m^2z\right) + Li_2\left(\frac{zx}{y}\right) - \frac{\pi^2}{3} \\ & + \log(v)\log\left(\frac{w}{m^2xz}\right) - \log x\log w - 2\log\left(m^2\right)\log z + \left(\log\left(m^2/w\right)\right)^2. \end{aligned}$$

- Hyperbolic volume

$$\operatorname{Im} V_{\mathcal{M}}(v,w,x,y,z;1) = D\left(\frac{w}{x}\right) + D\left(\frac{vw}{y}\right) - D\left(\frac{vx}{y}\right) + D\left(\frac{x}{wy}\right) - D(z) + D\left(\frac{zx}{y}\right)$$

$$\begin{aligned} V_{\mathcal{M}}(v,w,x,y,z;1) &= L\left(\frac{w}{x}\right) + L\left(\frac{vw}{y}\right) - L\left(\frac{vx}{y}\right) + L\left(\frac{x}{wy}\right) \\ &\quad - L(z) + L\left(\frac{zx}{y}\right) - \frac{\pi^2}{3} - \pi\mathrm{i}\log\left(\frac{v^2}{w}\right) \\ &= -2\pi^2\cdot 0.478273\ldots + \mathrm{i}\cdot 5.13794\ldots \end{aligned}$$

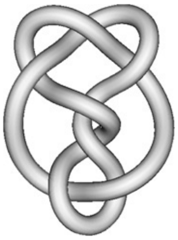
with

$$\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1.10278 + 0.665457\mathrm{i} \\ -0.102785 + 0.665457\mathrm{i} \\ 1.0 \\ -0.664742 - 0.401127\mathrm{i} \\ -0.226699 - 1.46771\mathrm{i} \end{pmatrix}.$$

- A-polynomial

$$\begin{pmatrix} 0 & 0 & -1 & 3 & -3 & 1 \\ 0 & 0 & 3 & -10 & 7 & 0 \\ 0 & 0 & -3 & 3 & 4 & 0 \\ 0 & 0 & -2 & 21 & -6 & 0 \\ 0 & 1 & 10 & -3 & 1 & 0 \\ 0 & -2 & 6 & -17 & 3 & 0 \\ 0 & 3 & -17 & 6 & -2 & 0 \\ 0 & 1 & -3 & 10 & 1 & 0 \\ 0 & -6 & 21 & -2 & 0 & 0 \\ 0 & 4 & 3 & -3 & 0 & 0 \\ 0 & 7 & -10 & 3 & 0 & 0 \\ 1 & -3 & 3 & -1 & 0 & 0 \end{pmatrix}.$$

A.1.7. 7_5



$$\begin{cases}\mathrm{Vol}(S^3 \setminus \mathcal{K}) = 6.443537\dots\\ \mathrm{cs}(S^3 \setminus \mathcal{K}) = 0.12055587\dots \bmod 1/2.\end{cases}$$

- Partition function

$$\begin{aligned}Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p\,\delta_C(p;u)&\langle p_2, p_{13}|S|p_1, p_{12}\rangle\langle p_4, p_1|S|p_3, p_{10}\rangle\\ &\times\langle p_6, p_3|S|p_2, p_{13}\rangle\langle p_5, p_{11}|S^{-1}|p_4, p_7\rangle\langle p_8, p_{12}|S^{-1}|p_6, p_5\rangle\\ &\times\langle p_7, p_9|S|p_8, p_{14}\rangle\langle p_{10}, p_{14}|S^{-1}|p_9, p_{11}\rangle.\end{aligned}$$

- Condition $\delta_C(p;u)$

$$p_3 + p_7 + p_{10} = p_1 + p_5 + p_{11} - 2u$$

- Potential function

$$\begin{aligned}V_{\mathcal{M}}(u,v,w,x,y,z;m) = {}&-Li_2\left(\frac{u}{v}\right) + Li_2\left(um^2\right) + Li_2\left(\frac{x}{m^2}\right)\\ &+Li_2\left(\frac{m^2}{y}\right) + Li_2\left(\frac{v}{m^2wy}\right) - Li_2\left(\frac{v}{z}\right) - Li_2\left(\frac{wx}{z}\right) - \frac{\pi^2}{6}\\ &+\log\left(m^2\right)\log\left(\frac{uw^2}{v}\right) + \log x\log y + \log u\log z.\end{aligned}$$

- Hyperbolic volume

$$\begin{aligned}\mathrm{Im}\,V_{\mathcal{M}}(u,v,w,x,y,z;m=1) &= -D\left(\frac{u}{v}\right) + D(u) + D(x) + D\left(\frac{1}{y}\right)\\ &\quad + D\left(\frac{v}{wy}\right) - D\left(\frac{v}{z}\right) - D\left(\frac{wx}{z}\right)\\ V_{\mathcal{M}}(u,v,w,x,y,z;m=1) &= -\frac{\pi^2}{6} - L\left(\frac{u}{v}\right) + L(u) + L(x) + L\left(\frac{1}{y}\right)\\ &\quad + L\left(\frac{v}{wy}\right) - L\left(\frac{v}{z}\right) - L\left(\frac{wx}{z}\right) - \pi\mathrm{i}\log u\\ &= 2\pi^2\cdot 0.120557\dots + \mathrm{i}\cdot 6.44354\dots\end{aligned}$$

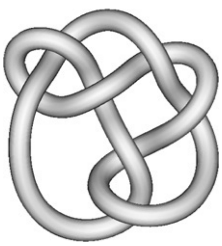
with

$$\begin{pmatrix}u\\v\\w\\x\\y\\z\end{pmatrix} = \begin{pmatrix}0.38762 + 1.0287\mathrm{i}\\ -0.572726 + 0.717749\mathrm{i}\\ -0.259819 + 0.832925\mathrm{i}\\ 0.18596 + 0.689115\mathrm{i}\\ 0.365014 - 1.35264\mathrm{i}\\ -0.679246 - 0.851242\mathrm{i}\end{pmatrix}.$$

- A-polynomial

$$
\begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -13 & -17 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 10 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 7 & 35 & 12 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & -32 & -23 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -56 & -6 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 24 & 48 & 6 & 0 & 0 & 0 & 0 \\
0 & 0 & 28 & 15 & -11 & 0 & 0 & 0 & 0 \\
0 & 0 & -22 & -82 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & -14 & -28 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 14 & 47 & 4 & -1 & 0 & 0 & 0 \\
0 & 0 & -3 & -13 & -11 & 7 & 0 & 0 & 0 \\
0 & 0 & 0 & -46 & 12 & -12 & 0 & 0 & 0 \\
0 & 0 & 0 & 15 & -16 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 15 & -52 & 15 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & -16 & 15 & 0 & 0 & 0 \\
0 & 0 & 0 & -12 & 12 & -46 & 0 & 0 & 0 \\
0 & 0 & 0 & 7 & -11 & -13 & -3 & 0 & 0 \\
0 & 0 & 0 & -1 & 4 & 47 & 14 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & -28 & -14 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & -82 & -22 & 0 & 0 \\
0 & 0 & 0 & 0 & -11 & 15 & 28 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 48 & 24 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -6 & -56 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -23 & -32 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & 12 & 35 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 10 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -17 & -13 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix}.
$$

A.1.8. 7_6



$$
\begin{cases}
\mathrm{Vol}(S^3 \setminus \mathcal{K}) = 7.08493\ldots \\
\mathrm{cs}(S^3 \setminus \mathcal{K}) = 0.18228319\ldots \bmod 1/2.
\end{cases}
$$

- Partition function

$$
\begin{aligned}
Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p\, \delta_C(p;u) &\langle p_1, p_{14}|S^{-1}|p_3, p_{13}\rangle \langle p_4, p_{13}|S^{-1}|p_1, p_2\rangle \langle p_9, p_5|S|p_4, p_{14}\rangle \\
&\times \langle p_3, p_2|S|p_6, p_{15}\rangle \langle p_7, p_6|S|p_5, p_{16}\rangle \langle p_{11}, p_{12}|S^{-1}|p_7, p_8\rangle \\
&\times \langle p_8, p_{15}|S^{-1}|p_6, p_{10}\rangle \langle p_{10}, p_{16}|S^{-1}|p_{11}, p_{12}\rangle .
\end{aligned}
$$

- Condition $\delta_C(p;u)$

$$
p_6 + p_{10} = p_5 + p_8 - 2u.
$$

- Potential function

$$
\begin{aligned}
V_{\mathcal{M}}(t,u,v,w,x,y,z;m) = &-Li_2(t) - Li_2\left(um^2\right) + Li_2\left(\frac{v}{x}\right) - Li_2\left(\frac{m^2}{x}\right) \\
&+ Li_2\left(\frac{t}{y}\right) + Li_2\left(\frac{u}{wy}\right) - Li_2\left(\frac{1}{m^2 z}\right) - Li_2\left(\frac{vw}{z}\right) + \frac{\pi^2}{3} \\
&- (\log(m^2))^2 + 2\log\left(m^2\right)\log\left(\frac{x}{yw}\right) + \log(t/v)\log(x/y) - \log u \log z .
\end{aligned}
$$

- Hyperbolic volume

$$
\begin{aligned}
\operatorname{Im} V_{\mathcal{M}}(t,u,v,w,x,y,z;1) &= -D(t) - D(u) + D\left(\frac{v}{x}\right) - D\left(\frac{1}{x}\right) + D\left(\frac{t}{y}\right) \\
&\quad + D\left(\frac{u}{wy}\right) - D\left(\frac{1}{z}\right) - D\left(\frac{vw}{z}\right) \\
V_{\mathcal{M}}(t,u,v,w,x,y,z;1) &= -L(t) - L(u) + L\left(\frac{v}{x}\right) - L\left(\frac{1}{x}\right) + L\left(\frac{t}{y}\right) \\
&\quad + L\left(\frac{u}{wy}\right) - L\left(\frac{1}{z}\right) - L\left(\frac{vw}{z}\right) + \frac{\pi^2}{3} - 2\pi \mathrm{i} \log t \\
&= -2\pi^2 \cdot 0.182283\ldots + \mathrm{i} \cdot 7.08493\ldots
\end{aligned}
$$

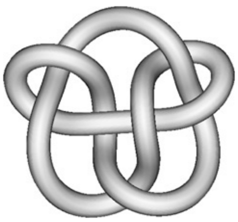
with

$$
\begin{pmatrix} t \\ u \\ v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.558614 - 1.43795\mathrm{i} \\ -0.0892864 - 0.842785\mathrm{i} \\ -0.280101 + 1.13004\mathrm{i} \\ 0.450985 - 0.808297\mathrm{i} \\ 0.234736 + 0.604244\mathrm{i} \\ -0.20665 - 0.833705\mathrm{i} \\ -0.12431 + 1.17337\mathrm{i} \end{pmatrix} .
$$

- A-polynomial

$$
\begin{pmatrix}
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -6 & 7 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -2 & 11 & -16 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 6 & 2 & 1 & -9 & 0 & 0 & 0 & 0 & 0\\
1 & -5 & -16 & 34 & 32 & 0 & 0 & 0 & 0 & 0\\
0 & -5 & -7 & 10 & -30 & 2 & 0 & 0 & 0 & 0\\
0 & 16 & 9 & -80 & -68 & -16 & 0 & 0 & 0 & 0\\
0 & 5 & 8 & -9 & 98 & 41 & 0 & 0 & 0 & 0\\
0 & -9 & 42 & 62 & 164 & -18 & 1 & 0 & 0 & 0\\
0 & 3 & 11 & 10 & -212 & -78 & 7 & 0 & 0 & 0\\
0 & 0 & -37 & 34 & -266 & 52 & 19 & 0 & 0 & 0\\
0 & 0 & 8 & 83 & 196 & 158 & -29 & 0 & 0 & 0\\
0 & 0 & 23 & -44 & 377 & -85 & 10 & 0 & 0 & 0\\
0 & 0 & -16 & -48 & 24 & -237 & 47 & 3 & 0 & 0\\
0 & 0 & 3 & 47 & -237 & 24 & -48 & -16 & 0 & 0\\
0 & 0 & 0 & 10 & -85 & 377 & -44 & 23 & 0 & 0\\
0 & 0 & 0 & -29 & 158 & 196 & 83 & 8 & 0 & 0\\
0 & 0 & 0 & 19 & 52 & -266 & 34 & -37 & 0 & 0\\
0 & 0 & 0 & 7 & -78 & -212 & 10 & 11 & 3 & 0\\
0 & 0 & 0 & 1 & -18 & 164 & 62 & 42 & -9 & 0\\
0 & 0 & 0 & 0 & 41 & 98 & -9 & 8 & 5 & 0\\
0 & 0 & 0 & 0 & -16 & -68 & -80 & 9 & 16 & 0\\
0 & 0 & 0 & 0 & 2 & -30 & 10 & -7 & -5 & 0\\
0 & 0 & 0 & 0 & 0 & 32 & 34 & -16 & -5 & 1\\
0 & 0 & 0 & 0 & 0 & -9 & 1 & 2 & 6 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & -16 & 11 & -2 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 7 & -6 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0
\end{pmatrix}.
$$

A.1.9. 7_7



$$
\begin{cases}
\mathrm{Vol}(S^3 \setminus \mathcal{K}) = 7.64338\ldots\\
\mathrm{cs}(S^3 \setminus \mathcal{K}) = 0.1329856\ldots \bmod 1/2.
\end{cases}
$$

- Partition function

$$
\begin{aligned}
Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p\,\delta_C(p;u) &\langle p_1, p_3|S|p_{15}, p_{16}\rangle\langle p_2, p_4|S|p_1, p_{14}\rangle\langle p_5, p_{13}|S^{-1}|p_8, p_4\rangle\\
&\times \langle p_6, p_{15}|S^{-1}|p_3, p_2\rangle\langle p_7, p_{16}|S^{-1}|p_5, p_6\rangle\langle p_{10}, p_{14}|S^{-1}|p_7, p_9\rangle\\
&\times \langle p_8, p_9|S|p_{11}, p_{12}\rangle\langle p_{11}, p_{12}|S|p_{10}, p_{13}\rangle.
\end{aligned}
$$

- Condition $\delta_C(p;u)$

$$
p_6 + p_{15} = p_2 + p_3 - 2u.
$$

- Potential function

$$\begin{aligned} V_{\mathcal{M}}(t,u,v,w,x,y,z;m) &= Li_2\left(vm^2\right) + Li_2\left(tm^2w\right) - Li_2\left(\frac{tm^2}{x}\right) - Li_2\left(\frac{1}{m^2wx}\right) \\ &\quad - Li_2(vx) + Li_2(ux) + Li_2\left(\frac{1}{m^4z}\right) - Li_2\left(\frac{u}{m^2z}\right) \\ &\quad + \log\left(m^2\right)\log\left(\frac{vu}{m^2wy}\right) + \log(tu)\log\left(\frac{wx}{y}\right) + \log v \log z. \end{aligned}$$

- Hyperbolic volume

$$\begin{aligned} \operatorname{Im} V_{\mathcal{M}}(t,u,v,w,x,y,z;1) &= D(v) + D(tw) - D\left(\frac{t}{x}\right) - D\left(\frac{1}{wx}\right) \\ &\quad - D(vx) + D(ux) + D\left(\frac{1}{z}\right) - D\left(\frac{u}{z}\right) \\ V_{\mathcal{M}}(t,u,v,w,x,y,z;1) &= L(v) + L(tw) - L\left(\frac{t}{x}\right) - L\left(\frac{1}{wx}\right) \\ &\quad - L(vx) + L(ux) + L\left(\frac{1}{z}\right) - L\left(\frac{u}{z}\right) + 2\pi \mathrm{i} \log t \\ &= -2\pi^2 \cdot 0.867014\ldots + \mathrm{i} \cdot 7.64338\ldots \end{aligned}$$

with

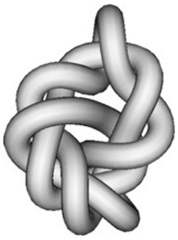
$$\begin{pmatrix} t \\ u \\ v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -0.899232 + 0.400532\mathrm{i} \\ -0.927958 - 0.413327\mathrm{i} \\ 0.0287264 + 0.813859\mathrm{i} \\ -0.351808 - 0.720342\mathrm{i} \\ -0.927958 - 0.413327\mathrm{i} \\ -0.927958 - 0.413327\mathrm{i} \\ 0.0433154 - 1.22719\mathrm{i} \end{pmatrix}.$$

- A-polynomial

$$\begin{pmatrix}
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -7 & 8 & 0 & 0 & 0 \\
0 & 0 & 0 & 15 & -19 & 1 & 0 & 0 \\
0 & 0 & 3 & 1 & -4 & -6 & 0 & 0 \\
0 & 0 & -18 & -30 & 59 & 11 & 0 & 0 \\
0 & 0 & 23 & -1 & 0 & -1 & 0 & 0 \\
0 & 3 & 27 & 41 & -123 & -8 & -2 & 0 \\
0 & -11 & -65 & 7 & -2 & 1 & 7 & 0 \\
0 & 4 & -19 & -29 & 130 & -28 & -7 & 0 \\
1 & 20 & 84 & -46 & 35 & 16 & -7 & 0 \\
0 & -7 & 16 & 35 & -46 & 84 & 20 & 1 \\
0 & -7 & -28 & 130 & -29 & -19 & 4 & 0 \\
0 & 7 & 1 & -2 & 7 & -65 & -11 & 0 \\
0 & -2 & -8 & -123 & 41 & 27 & 3 & 0 \\
0 & 0 & -1 & 0 & -1 & 23 & 0 & 0 \\
0 & 0 & 11 & 59 & -30 & -18 & 0 & 0 \\
0 & 0 & -6 & -4 & 1 & 3 & 0 & 0 \\
0 & 0 & 1 & -19 & 15 & 0 & 0 & 0 \\
0 & 0 & 0 & 8 & -7 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0
\end{pmatrix}.$$

A.2. Complement of "simple" hyperbolic knots

A.2.1. $K4_4$



$$\begin{cases}\mathrm{Vol}(S^3 \setminus \mathcal{K}) = 3.6086890618\ldots \\ \mathrm{cs}(S^3 \setminus \mathcal{K}) = 0.1139831647\ldots \bmod 1/2.\end{cases}$$

- Partition function

$$Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p\, \delta_C(p;u) \langle p_3, p_2|S^{-1}|p_2, p_1\rangle \langle p_{12}, p_5|S|p_{10}, p_3\rangle \langle p_6, p_1|S^{-1}|p_{11}, p_{12}\rangle \times \langle p_{10}, p_9|S|p_8, p_6\rangle \langle p_4, p_7|S|p_9, p_4\rangle \langle p_{11}, p_8|S|p_7, p_5\rangle.$$

- Condition $\delta_C(p;u)$

$$p_{12} - p_1 + p_5 - p_8 - p_6 + p_9 = -2u.$$

- Potential function

$$\begin{aligned} V_{\mathcal{M}}(v,w,x,y,z;m) = & -Li_2\left(\frac{x}{w}\right) + Li_2\left(\frac{1}{vm^2y}\right) - Li_2\left(\frac{z}{vm^2}\right) \\ & + Li_2\left(m^2xy\right) + Li_2\left(\frac{m^2}{yz}\right) + Li_2\left(\frac{wy}{z}\right) - \frac{\pi^2}{3} + \log\left(\frac{v}{z}\right)\log\left(m^4xy\right) \\ & + \log(x)\log\left(\frac{m^2w^2}{x}\right) + 2\log(wy)\log(y) + \log\left(m^2\right)\log\left(m^4w^3\right). \end{aligned}$$

- Hyperbolic volume

$$\mathrm{Im}\, V_{\mathcal{M}}(v,w,x,y,z;1) = -D\left(\frac{x}{w}\right) + D\left(\frac{1}{vy}\right) - D\left(\frac{z}{v}\right) + D(xy) + D\left(\frac{1}{yz}\right) + D\left(\frac{wy}{z}\right)$$

$$\begin{aligned} V_{\mathcal{M}}(v,w,x,y,z;1) &= -\frac{\pi^2}{3} - L\left(\frac{x}{w}\right) + L\left(\frac{1}{vy}\right) - L\left(\frac{z}{v}\right) \\ &\quad + L(xy) + L\left(\frac{1}{yz}\right) + L\left(\frac{wy}{z}\right) - \pi\mathrm{i}\log(x/y) \\ &= -2\pi^2 \cdot 0.113983\ldots + \mathrm{i}\cdot 3.60869\ldots \end{aligned}$$

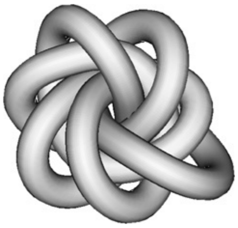
with

$$\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -0.06796 - 1.03267\mathrm{i} \\ -0.597112 + 0.762045\mathrm{i} \\ 1.29516 + 0.539127\mathrm{i} \\ 0.457778 + 1.02559\mathrm{i} \\ -0.396648 - 0.345221\mathrm{i} \end{pmatrix}.$$

- A-polynomial

$$\begin{aligned} A(\ell,m) = & 1 + (m^{30} - 2m^{32} + m^{34})\ell + (-m^{58} + 2m^{60} - 10m^{62} + 4m^{64} - m^{66})\ell^2 \\ & + (-2m^{90} + 3m^{92} - m^{96})\ell^3 + (m^{120} + 8m^{122} + 6m^{124})\ell^4 \\ & + (m^{150} - m^{152} - m^{154} + m^{156})\ell^5 + (2m^{180} - 12m^{182} - 12m^{186} + 2m^{188})\ell^6 \\ & + (m^{212} - m^{214} - m^{216} + m^{218})\ell^7 + (6m^{244} + 8m^{246} + m^{248})\ell^8 \\ & + (-m^{272} + 3m^{276} - 2m^{278})\ell^9 + (-m^{302} + 4m^{304} - 10m^{406} + 2m^{408} - m^{310})\ell^{10} \\ & + (m^{334} - 2m^{336} + m^{338})\ell^{11} + m^{368}\ell^{12}. \end{aligned}$$

A.2.2. $K5_1$



$$\begin{cases}\mathrm{Vol}(S^3 \setminus \mathcal{K}) = 3.4179148372\ldots\\ \mathrm{cs}(S^3 \setminus \mathcal{K}) = 0.1517274811\ldots \bmod 1/2.\end{cases}$$

- Partition function

$$Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p\,\delta_C(p;u) \langle p_5, p_2|S^{-1}|p_1, p_3\rangle \langle p_1, p_9|S^{-1}|p_9, p_2\rangle \langle p_4, p_3|S^{-1}|p_5, p_7\rangle \\ \times \langle p_{10}, p_6|S^{-1}|p_4, p_{11}\rangle \langle p_{12}, p_7|S^{-1}|p_6, p_{10}\rangle \langle p_{11}, p_8|S^{-1}|p_8, p_{12}\rangle.$$

- Condition $\delta_C(p;u)$

$$p_{11} - p_6 - p_{10} + p_7 = -2u.$$

- Potential function

$$\begin{aligned} V_{\mathcal{M}}(v,w,x,y,z;m) = & -Li_2\left(\frac{w}{x^2}\right) - Li_2\left(\frac{wx}{y}\right) - Li_2\left(\frac{y^2}{m^4 vz}\right) - Li_2\left(\frac{y^2}{wz}\right) \\ & - Li_2\left(\frac{z}{vy}\right) - Li_2\left(\frac{m^2 z}{vy}\right) + \pi^2 + 5\log\left(m^2\right)\log\left(\frac{y}{z}\right) \\ & + 2\log w \log\left(\frac{xy}{w}\right) + 2\log z\log\left(\frac{y^3}{z}\right) - 2\left(\log(m^2)\right)^2 - 2(\log x)^2 - 5(\log y)^2. \end{aligned}$$

- Hyperbolic volume

$$\operatorname{Im} V_{\mathcal{M}}(v,w,x,y,z;1) = -D\left(\frac{w}{x^2}\right) - D\left(\frac{wx}{y}\right) - D\left(\frac{y^2}{vz}\right) - D\left(\frac{y^2}{wz}\right) - 2D\left(\frac{z}{vy}\right)$$

$$\begin{aligned} V_{\mathcal{M}}(v,w,x,y,z;1) = & -L\left(\frac{w}{x^2}\right) - L\left(\frac{wx}{y}\right) - L\left(\frac{y^2}{vz}\right) \\ & - L\left(\frac{y^2}{wz}\right) - 2L\left(\frac{z}{vy}\right) + \pi^2 - \pi \mathrm{i}\log\left(\frac{y}{v}\right) \\ = & -2\pi^2 \cdot 0.151727\ldots + \mathrm{i}\cdot 3.41791\ldots \end{aligned}$$

with

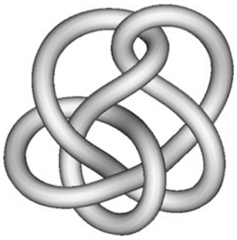
$$\begin{pmatrix} v\\ w\\ x\\ y\\ z\end{pmatrix} = \begin{pmatrix} 0.465534 - 0.473866\mathrm{i}\\ 0.693244 + 0.159750\mathrm{i}\\ -1.085877 - 0.175545\mathrm{i}\\ -0.952444 - 0.928780\mathrm{i}\\ -0.907927 + 0.840443\mathrm{i}\end{pmatrix}.$$

- A-polynomial

$$\begin{aligned} A_{\mathcal{K}}(\ell,m) = & -1 + \left(-m^{32} + m^{34}\right)\ell + \left(9m^{64} - 3m^{66} + m^{68}\right)\ell^2 \\ & + \left(m^{92} - 3m^{94} + 12m^{96} - 14m^{98} + 5m^{100} - m^{102}\right)\ell^3 \\ & + \left(m^{124} - 7m^{126} - 18m^{128} + 5m^{130} - 2m^{132}\right)\ell^4 \\ & + \left(-m^{156} - 7m^{158} + 2m^{160} + 6m^{162}\right)\ell^5 \end{aligned}$$

$$
\begin{aligned}
&+ \left(-m^{188} + 17m^{190} + 20m^{192} - 2m^{194} + m^{196}\right) \ell^6 \\
&+ \left(-2m^{218} + 14m^{220} - 12m^{222} + 12m^{224} - 14m^{226} + 2m^{228}\right) \ell^7 \\
&+ \left(-m^{250} + 2m^{252} - 20m^{254} - 17m^{256} + m^{258}\right) \ell^8 \\
&+ \left(-6m^{284} - 2m^{286} + 7m^{288} + m^{290}\right) \ell^9 \\
&+ \left(2m^{314} - 5m^{316} + 18m^{318} + 7m^{320} - m^{322}\right) \ell^{10} \\
&+ \left(m^{344} - 5m^{346} + 14m^{348} - 12m^{350} + 3m^{352} - m^{354}\right) \ell^{11} \\
&+ \left(-m^{378} + 3m^{380} - 9m^{382}\right) \ell^{12} + \left(-m^{412} + m^{414}\right) \ell^{13} + \ell^{14} m^{446}.
\end{aligned}
$$

A.2.3. $K5_9$ or 10_{132}



$$
\begin{cases}
\mathrm{Vol}(S^3 \setminus \mathcal{K}) = 4.0568602242\ldots \\
\mathrm{cs}(S^3 \setminus \mathcal{K}) = 0.1867489858\ldots \bmod 1/2.
\end{cases}
$$

- Partition function

$$
\begin{aligned}
Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p\, \delta_C(p; u) \langle p_1, p_9 | S^{-1} | p_2, p_3 \rangle \langle p_4, p_2 | S^{-1} | p_9, p_5 \rangle \\
\times \langle p_6, p_7 | S^{-1} | p_7, p_8 \rangle \langle p_5, p_3 | S | p_6, p_{10} \rangle \langle p_{10}, p_8 | S | p_4, p_1 \rangle.
\end{aligned}
$$

- Condition $\delta_C(p; u)$

$$
p_2 + p_8 + p_{10} = p_1 + p_5 + p_9 - 2u.
$$

- Potential function

$$
\begin{aligned}
V_{\mathcal{M}}(w, x, y, z; m) = &\, Li_2(w) - Li_2\left(\frac{w}{y}\right) - Li_2\left(\frac{y}{x}\right) - Li_2(z) + Li_2\left(\frac{yz}{m^2}\right) + \frac{\pi^2}{6} \\
&+ \left(\log\left(m^2\right)\right)^2 - \log\left(m^2\right) \log(xyw) + \log w \log\left(\frac{y}{z}\right) + 2 \log y \log(xz).
\end{aligned}
$$

- Hyperbolic volume

$$
\operatorname{Im} V_{\mathcal{M}}(w, x, y, z; 1) = D(w) - D\left(\frac{w}{y}\right) - D\left(\frac{y}{x}\right) - D(z) + D(yz)
$$

$$
\begin{aligned}
V_{\mathcal{M}}(w, x, y, z; 1) &= \frac{\pi^2}{6} + L(w) - L\left(\frac{w}{y}\right) - L\left(\frac{y}{x}\right) - L(z) + L(yz) - \pi \mathrm{i} \log(wxyz) \\
&= 2\pi^2 \cdot 0.186749\ldots + \mathrm{i} \cdot 4.05686\ldots
\end{aligned}
$$

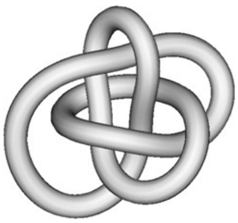
with

$$
\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -0.0498076 + 0.754729\mathrm{i} \\ -1.54094 - 1.35872\mathrm{i} \\ -0.821578 - 0.131699\mathrm{i} \\ -0.0844626 - 0.905094\mathrm{i} \end{pmatrix}.
$$

- A-polynomial

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -7 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 3 & 7 & -4 & -2 & 0 & 0 \\
0 & 1 & 4 & 2 & 13 & 4 & -6 & -3 & -1 \\
0 & -1 & -5 & -7 & -6 & 5 & 13 & 1 & 0 \\
0 & 0 & -5 & -12 & -18 & 9 & 9 & 4 & 0 \\
0 & 0 & 12 & 13 & -15 & -10 & -7 & -4 & 0 \\
0 & 4 & 7 & 10 & 15 & -13 & -12 & 0 & 0 \\
0 & -4 & -9 & -9 & 18 & 12 & 5 & 0 & 0 \\
0 & -1 & -13 & -5 & 6 & 7 & 5 & 1 & 0 \\
1 & 3 & 6 & -4 & -13 & -2 & -4 & -1 & 0 \\
0 & 0 & 2 & 4 & -7 & -3 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 7 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

A.2.4. $K5_{12}$ or 8_{20}



$$\begin{cases} \mathrm{Vol}(S^3 \setminus \mathcal{K}) = 4.1249032518\ldots \\ \mathrm{cs}(S^3 \setminus \mathcal{K}) = 0.1033634474\ldots \bmod 1/2. \end{cases}$$

- Partition function

$$Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p\, \delta_C(p; u) \langle p_8, p_2 | S | p_1, p_6 \rangle \langle p_1, p_{10} | S^{-1} | p_{10}, p_4 \rangle \\ \times \langle p_9, p_3 | S | p_2, p_9 \rangle \langle p_4, p_7 | S^{-1} | p_5, p_3 \rangle \langle p_6, p_5 | S^{-1} | p_7, p_8 \rangle.$$

- Condition $\delta_C(p; u)$

$$p_3 - p_7 + p_6 - p_2 - p_8 + p_5 = -2u.$$

- Potential function

$$\begin{aligned} V_{\mathcal{M}}(w, x, y, z; m) = & -Li_2\left(\frac{x}{m^2 w}\right) - Li_2\left(\frac{m^4 w}{xy}\right) + Li_2\left(\frac{x}{y}\right) - Li_2\left(\frac{z}{wy}\right) \\ & + Li_2\left(\frac{z}{xy}\right) + \frac{\pi^2}{6} - 2\left(\log\left(m^2\right)\right)^2 - 2(\log w)^2 + (\log x)^2 \\ & + \log\left(m^2\right) \log\left(\frac{xz^2}{w^5}\right) + 2\log w \log x + 2\log w \log z - 2\log x \log z. \end{aligned}$$

- Hyperbolic volume

$$\mathrm{Im}\, V_{\mathcal{M}}(w, x, y, z; 1) = -D\left(\frac{x}{w}\right) - D\left(\frac{w}{xy}\right) + D\left(\frac{x}{y}\right) - D\left(\frac{z}{wy}\right) + D\left(\frac{z}{xy}\right)$$

$$\begin{aligned} V_{\mathcal{M}}(w, x, y, z; 1) &= -L\left(\frac{x}{w}\right) - L\left(\frac{w}{xy}\right) + L\left(\frac{x}{y}\right) - L\left(\frac{z}{wy}\right) + L\left(\frac{z}{xy}\right) + \frac{\pi^2}{6} + \pi \mathrm{i} \log\left(\frac{y}{z}\right) \\ &= -2\pi^2 \cdot 0.396637\ldots + \mathrm{i} \cdot 4.1249\ldots \end{aligned}$$

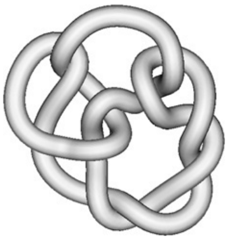
with

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -0.723387 - 0.90034\mathrm{i} \\ -0.637406 + 0.318768\mathrm{i} \\ -0.483596 + 0.741071\mathrm{i} \\ 1.08906 - 0.727199\mathrm{i} \end{pmatrix}.$$

- A-polynomial

$$\begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 & 0 \\ 0 & -2 & 0 & -2 & 0 & 0 \\ 0 & 2 & 3 & -4 & -1 & 0 \\ 0 & 1 & -3 & 0 & -5 & -1 \\ -1 & -5 & 0 & -3 & 1 & 0 \\ 0 & -1 & -4 & 3 & 2 & 0 \\ 0 & 0 & -2 & 0 & -2 & 0 \\ 0 & 0 & 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

A.2.5. $K5_{13}$



$$\begin{cases} \mathrm{Vol}(S^3 \setminus \mathcal{K}) = 4.1249032518\ldots \\ \mathrm{cs}(S^3 \setminus \mathcal{K}) = 0.0200301140\ldots \bmod 1/2. \end{cases}$$

- Partition function

$$Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p\,\delta_C(p;u) \langle p_2, p_9 | S^{-1} | p_1, p_3 \rangle \langle p_7, p_6 | S^{-1} | p_6, p_2 \rangle \times \langle p_4, p_5 | S^{-1} | p_9, p_7 \rangle \langle p_8, p_1 | S | p_4, p_{10} \rangle \langle p_{10}, p_3 | S | p_5, p_8 \rangle. \tag{A.2}$$

- Condition $\delta_C(p;u)$

$$p_1 + p_5 + p_8 = p_7 + p_9 + p_{10} - 2u.$$

- Potential function

$$V_{\mathcal{M}}(w,x,y,z;m) = -Li_2(mx) + Li_2\left(\frac{x}{w}\right) - Li_2\left(\frac{1}{w^2 y}\right) - Li_2\left(\frac{z}{y}\right) + Li_2\left(\frac{z}{wy}\right) + \frac{\pi^2}{6} - \left(\log\left(m^2\right)\right)^2 - \log\left(m^2\right)\log(wz) + \log x \log\left(\frac{z}{yw}\right) - 2\log w \log z.$$

- Hyperbolic volume

$$\mathrm{Im}\, V_{\mathcal{M}}(w,x,y,z;1) = -D(x) + D\left(\frac{x}{w}\right) - D\left(\frac{1}{w^2 y}\right) - D\left(\frac{z}{y}\right) + D\left(\frac{z}{wy}\right)$$

$$
\begin{aligned}
V_{\mathcal{M}}(w,x,y,z;1) &= \frac{\pi^2}{6} - L(x) + L\left(\frac{x}{w}\right) - L\left(\frac{1}{w^2 y}\right) \\
&\quad - L\left(\frac{z}{y}\right) + L\left(\frac{z}{wy}\right) - \pi \mathrm{i} \log\left(\frac{xz}{w}\right) \\
&= -2\pi^2 \cdot 0.47997\ldots + \mathrm{i} \cdot 4.1249\ldots
\end{aligned}
$$

with

$$
\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -0.812447 + 0.173142\mathrm{i} \\ -0.0890598 - 0.727199\mathrm{i} \\ -1.71268 + 1.30259\mathrm{i} \\ 1.09977 + 1.12945\mathrm{i} \end{pmatrix}.
$$

- A-polynomial

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -10 & 1 & 3 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 8 & -14 & -11 & 8 & 0 & 0 & 0 & 0 \\
0 & -5 & 11 & 5 & -1 & -12 & 0 & 0 & 0 & 0 \\
-1 & 5 & -10 & 14 & 29 & -12 & 1 & 0 & 0 & 0 \\
0 & 0 & -5 & 4 & 5 & 16 & -1 & 0 & 0 & 0 \\
0 & -1 & 9 & -13 & -19 & 25 & -7 & 2 & 0 & 0 \\
0 & 0 & 5 & 3 & -33 & -3 & 9 & -7 & 0 & 0 \\
0 & 0 & -7 & 9 & -3 & -33 & 3 & 5 & 0 & 0 \\
0 & 0 & 2 & -7 & 25 & -19 & -13 & 9 & -1 & 0 \\
0 & 0 & 0 & -1 & 16 & 5 & 4 & -5 & 0 & 0 \\
0 & 0 & 0 & 1 & -12 & 29 & 14 & -10 & 5 & -1 \\
0 & 0 & 0 & 0 & -12 & -1 & 5 & 11 & -5 & 0 \\
0 & 0 & 0 & 0 & 8 & -11 & -14 & 8 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 3 & 1 & -10 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.
$$

A.2.6. $K5_{21}$ or 9_{46}

$$
\begin{cases} \mathrm{Vol}(S^3 \setminus \mathcal{K}) = 4.7517019655\ldots \\ \mathrm{cs}(S^3 \setminus \mathcal{K}) = 0.1450479602\ldots. \end{cases}
$$

- Partition function

$$
\begin{aligned}
Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p\, \delta_C(p;u) \langle p_4, p_1|S|p_6, p_7\rangle \langle p_2, p_9|S^{-1}|p_1, p_8\rangle \\
\times \langle p_7, p_8|S^{-1}|p_3, p_4\rangle \langle p_6, p_5|S^{-1}|p_9, p_{10}\rangle \langle p_3, p_{10}|S^{-1}|p_5, p_2\rangle.
\end{aligned}
$$

- Condition $\delta_C(p;u)$

$$
p_1 + p_4 + p_5 = p_7 + p_9 + p_{10} - 2u.
$$

- Potential function

$$V_{\mathcal{M}}(w,x,y,z;m) = -Li_2\left(\frac{w^2x}{m^2}\right) + Li_2\left(\frac{m^2y}{x}\right) - Li_2\left(\frac{m^2}{w^2xz}\right) - Li_2\left(\frac{z}{w}\right) - Li_2\left(\frac{wyz}{m^2}\right) + \frac{\pi^2}{2} + \left(\log\left(m^2\right)\right)^2 - 2(\log w)^2 - (\log z)^2 + \log\left(m^2\right)\log\left(\frac{yz}{x}\right) - \log w \log x - \log(yz)\log(xw).$$

- Hyperbolic volume

$$\operatorname{Im} V_{\mathcal{M}}(w,x,y,z;1) = -D\left(w^2x\right) + D\left(\frac{y}{x}\right) - D\left(\frac{1}{w^2xz}\right) - D\left(\frac{z}{w}\right) - D(wyz)$$

$$V_{\mathcal{M}}(w,x,y,z;1) = \frac{\pi^2}{2} - L\left(w^2x\right) + L\left(\frac{y}{x}\right) - L\left(\frac{1}{w^2xz}\right) - L\left(\frac{z}{w}\right) - L(wyz) = 2\pi^2\cdot 0.145048\ldots + \mathrm{i}\cdot 4.7517\ldots.$$

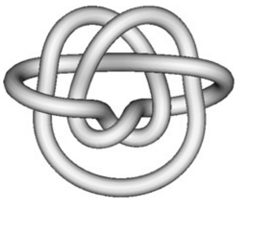
with

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1.21844 + 0.168108\mathrm{i} \\ 0.640448 - 0.637204\mathrm{i} \\ 1.0 \\ -0.445837 + 0.526085\mathrm{i} \end{pmatrix}.$$

- A-polynomial

$$\begin{pmatrix} 0 & 0 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 \\ -1 & -5 & -3 & 2 & 0 \\ 0 & 0 & -5 & -5 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & -5 & -5 & 0 & 0 \\ 0 & 2 & -3 & -5 & -1 \\ 0 & -1 & 5 & 2 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix}.$$

A.2.7. $K5_{22}$ or 10_{139}



$$\begin{cases} \mathrm{Vol}(S^3 \setminus \mathcal{K}) = 4.8511707573\ldots \\ \mathrm{cs}(S^3 \setminus \mathcal{K}) = 0.2289275614\ldots \bmod 1/2. \end{cases}$$

- Partition function

$$Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p\, \delta_C(p;u) \langle p_5, p_9|S|p_8, p_1\rangle \langle p_1, p_3|S|p_2, p_9\rangle \times \langle p_6, p_7|S|p_4, p_3\rangle \langle p_2, p_8|S|p_7, p_{10}\rangle \langle p_4, p_{10}|S|p_5, p_6\rangle.$$

- Condition $\delta_C(p;u)$

$$p_7 + p_9 + p_{10} = p_1 + p_3 + p_8 - 2u.$$

- Potential function

$$V_{\mathcal{M}}(w,x,y,z;m) = Li_2\left(\frac{1}{m^2 x}\right) + Li_2\left(\frac{m^4 w}{y}\right) + Li_2\left(\frac{w}{xz}\right) + Li_2\left(\frac{z}{w}\right) + Li_2\left(\frac{wz}{y}\right) - \frac{5\pi^2}{6} + 2\left(\log\left(m^2\right)\right)^2 + \log\left(m^2\right)\log\left(\frac{x}{y^2 z}\right) + \log x \log\left(\frac{z}{y}\right) + \log w \log\left(\frac{w}{z}\right).$$

- Hyperbolic volume

$$\operatorname{Im} V_{\mathcal{M}}(w,x,y,z;1) = D\left(\frac{1}{x}\right) + D\left(\frac{w}{y}\right) + D\left(\frac{w}{xz}\right) + D\left(\frac{z}{w}\right) + D\left(\frac{wz}{y}\right)$$

$$V_{\mathcal{M}}(w,x,y,z;1) = -\frac{5\pi^2}{6} + L\left(\frac{1}{x}\right) + L\left(\frac{w}{y}\right) + L\left(\frac{w}{xz}\right) + L\left(\frac{z}{w}\right) + L\left(\frac{wz}{y}\right) = -2\pi^2 \cdot 0.228928\ldots + \mathrm{i}\cdot 4.85117\ldots$$

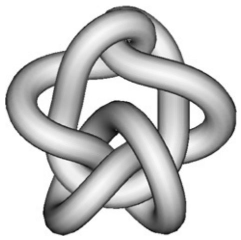
with

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.660443 - 0.716885\mathrm{i} \\ -0.0777392 - 0.946923\mathrm{i} \\ -0.460355 - 1.13932\mathrm{i} \\ 1.0 \end{pmatrix}.$$

- A-polynomial

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 7 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -3 & 0 \\
0 & 0 & 0 & 7 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

A.2.8. $K6_{10}$



$$\begin{cases}\mathrm{Vol}(S^3 \setminus \mathcal{K}) = 4.40083252\ldots \\ \mathrm{cs}(S^3 \setminus \mathcal{K}) = 0.21417417\ldots \bmod 1/2.\end{cases}$$

- Partition function

$$Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p\,\delta_C(p;u)\langle p_2, p_6|S^{-1}|p_4, p_1\rangle\langle p_1, p_{12}|S^{-1}|p_{10}, p_3\rangle\langle p_3, p_8|S|p_6, p_{11}\rangle \\ \times \langle p_7, p_9|S^{-1}|p_9, p_8\rangle\langle p_{10}, p_4|S^{-1}|p_{12}, p_5\rangle\langle p_{11}, p_5|S|p_2, p_7\rangle.$$

- Condition $\delta_C(p;u)$

$$p_5 + p_6 + p_{11} = p_1 + p_4 + p_8 - 2u.$$

- Potential function

$$\begin{aligned} V_{\mathcal{M}}(v,w,x,y,z;m) = & -Li_2\left(\frac{w}{v}\right) - Li_2\left(\frac{m^4 v}{y}\right) + Li_2\left(\frac{x}{y}\right) - Li_2\left(\frac{x}{z^2}\right) \\ & + Li_2\left(\frac{w}{z}\right) - Li_2\left(\frac{z}{m^2 v}\right) + \frac{\pi^2}{3} - 2\left(\log\left(m^2\right)\right)^2 \\ & + \log\left(m^2\right)\log\left(\frac{y^2 z}{v^3 w}\right) + \log v \log\left(\frac{yz}{v}\right) + \log x \log\left(\frac{wz^3}{x}\right) \\ & - \log z \log\left(yz^2 w\right). \end{aligned}$$

- Hyperbolic volume

$$\mathrm{Im}\, V_{\mathcal{M}}(v,w,x,y,z;1) = -D\left(\frac{w}{v}\right) - D\left(\frac{v}{y}\right) + D\left(\frac{x}{y}\right) - D\left(\frac{x}{z^2}\right) + D\left(\frac{w}{z}\right) - D\left(\frac{z}{v}\right)$$

$$\begin{aligned} V_{\mathcal{M}}(v,w,x,y,z;1) = & \frac{\pi^2}{3} - L\left(\frac{w}{v}\right) - L\left(\frac{v}{y}\right) + L\left(\frac{x}{y}\right) \\ & - L\left(\frac{x}{z^2}\right) + L\left(\frac{w}{z}\right) - L\left(\frac{z}{v}\right) - \pi\mathrm{i}\log\left(\frac{x}{z}\right) \\ = & -2\pi^2 \cdot 0.214174\ldots + \mathrm{i}\cdot 4.40083\ldots \end{aligned}$$

with

$$\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1.18608 + 0.874646\mathrm{i} \\ 1.0 \\ -1.09737 + 0.230836\mathrm{i} \\ -1.23271 + 1.09381\mathrm{i} \\ 0.40897 - 0.337176\mathrm{i} \end{pmatrix}.$$

- A-polynomial

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 10 & -7 & 3 & -1 & 1 & 0\\
0 & 0 & 0 & 0 & 1 & -24 & 16 & -3 & 6 & -2 & 1\\
0 & 0 & 0 & 2 & -4 & -5 & 5 & -19 & 1 & 1 & 0\\
0 & 0 & 0 & -10 & 7 & 47 & -39 & 13 & -12 & -6 & 0\\
0 & 0 & -1 & 15 & 5 & 18 & -14 & 22 & -5 & 6 & 0\\
0 & -1 & 3 & 3 & -36 & -21 & 45 & 12 & 9 & 0 & 0\\
0 & 1 & 7 & -35 & 16 & -69 & 30 & 2 & -6 & 0 & 0\\
0 & 0 & -19 & 15 & 44 & -45 & 6 & -40 & -7 & 0 & 0\\
0 & 0 & 7 & 40 & -6 & 45 & -44 & -15 & 19 & 0 & 0\\
0 & 0 & 6 & -2 & -30 & 69 & -16 & 35 & -7 & -1 & 0\\
0 & 0 & -9 & -12 & -45 & 21 & 36 & -3 & -3 & 1 & 0\\
0 & -6 & 5 & -22 & 14 & -18 & -5 & -15 & 1 & 0 & 0\\
0 & 6 & 12 & -13 & 39 & -47 & -7 & 10 & 0 & 0 & 0\\
0 & -1 & -1 & 19 & -5 & 5 & 4 & -2 & 0 & 0 & 0\\
-1 & 2 & -6 & 3 & -16 & 24 & -1 & 0 & 0 & 0 & 0\\
0 & -1 & 1 & -3 & 7 & -10 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

A.2.9. $K6_{22}$

$$\begin{cases}\mathrm{Vol}(S^3 \setminus \mathcal{K}) = 4.76988960 \cdots \\ \mathrm{cs}(S^3 \setminus \mathcal{K}) = 0.07253540 \ldots \bmod 1/2.\end{cases}$$

- Partition function

$$Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p \delta_C(p; u) \langle p_6, p_1 | S^{-1} | p_1, p_2 \rangle \langle p_2, p_{11} | S^{-1} | p_3, p_7 \rangle \langle p_{10}, p_7 | S | p_{12}, p_{10} \rangle \\ \times \langle p_3, p_9 | S^{-1} | p_4, p_{11} \rangle \langle p_5, p_4 | S | p_9, p_8 \rangle \langle p_8, p_{12} | S | p_6, p_5 \rangle.$$

- Condition $\delta_C(p; u)$

$$p_4 + p_5 + p_7 = p_8 + p_9 + p_{12} - 2u.$$

- Potential function

$$V_{\mathcal{M}}(v, w, x, y, z; m) = -Li_2\left(\frac{w}{v^2}\right) - Li_2\left(\frac{m^2 v}{w^2 x^2}\right) + Li_2\left(\frac{m^2 v}{y}\right) + Li_2\left(\frac{y}{v^2 x}\right) \\ - Li_2\left(\frac{vxz}{w}\right) + Li_2\left(\frac{xz}{y}\right) - \left(\log\left(m^2\right)\right)^2 + \log v \log\left(wz^2\right) - 3(\log w)^2 \\ + \log\left(m^2\right) \log\left(\frac{w^3 x^4 z^2}{y}\right) - 2 \log y \log\left(v^2 x^2 y z\right) + \log x \log\left(\frac{v^8}{w^4 x}\right).$$

- Hyperbolic volume

$$\mathrm{Im}\, V_{\mathcal{M}}(v, w, x, y, z; 1) = -D\left(\frac{w}{v^2}\right) - D\left(\frac{v}{w^2 x^2}\right) + D\left(\frac{v}{y}\right) + D\left(\frac{y}{v^2 x}\right) - D\left(\frac{vxz}{w}\right) + D\left(\frac{xz}{y}\right)$$

$$
\begin{aligned}
V_{\mathcal{M}}(v, w, x, y, z; 1) &= -L\left(\frac{w}{v^2}\right) - L\left(\frac{v}{w^2 x^2}\right) + L\left(\frac{v}{y}\right) \\
&\quad + L\left(\frac{y}{v^2 x}\right) - L\left(\frac{vxz}{w}\right) + L\left(\frac{xz}{y}\right) + \pi \mathrm{i} \log(vx) \\
&= 2\pi^2 \cdot 0.427465\ldots + \mathrm{i} \cdot 4.76989\ldots
\end{aligned}
$$

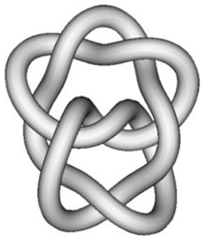
with

$$
\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.127216 + 0.708358\mathrm{i} \\ -0.534645 + 1.01011\mathrm{i} \\ -0.40328 + 0.755585\mathrm{i} \\ 0.66186 - 0.301752\mathrm{i} \\ -0.54977 - 1.03005\mathrm{i} \end{pmatrix}.
$$

- *A*-polynomial

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & -8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -16 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 19 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 20 & -13 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -44 & -5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 20 & -4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 30 & 8 & 9 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -56 & 2 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -5 & -10 & -36 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 35 & 30 & 22 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -32 & 69 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -9 & -41 & -65 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 16 & -13 & -9 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 7 & 126 & 7 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -9 & -13 & 16 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & -65 & -41 & -9 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 69 & -32 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 22 & 30 & 35 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -36 & -10 & -5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 2 & -56 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 9 & 8 & 30 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -4 & 20 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -5 & -44 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -13 & 20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 19 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & -16 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -8 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

A.2.10. $K6_{33}$ or 10_{140}



$$\begin{cases}\mathrm{Vol}(S^3 \setminus \mathcal{K}) = 5.21256682\dots\\ \mathrm{cs}(S^3 \setminus \mathcal{K}) = 0.10336001\dots \bmod 1/2.\end{cases}$$

- Partition function

$$Z_\gamma(\mathcal{M}_u) = \int_{\mathbb{R}} \mathrm{d}p\,\delta_C(p;u)\langle p_1, p_7|S|p_5, p_3\rangle\langle p_2, p_9|S^{-1}|p_1, p_4\rangle\langle p_3, p_{12}|S^{-1}|p_{10}, p_2\rangle \times \langle p_5, p_4|S^{-1}|p_6, p_{11}\rangle\langle p_8, p_6|S^{-1}|p_7, p_{12}\rangle\langle p_{10}, p_{11}|S^{-1}|p_9, p_8\rangle.$$

- Condition $\delta_C(p;u)$

$$p_3 + p_6 + p_{11} = p_2 + p_7 + p_9 - 2u.$$

- Potential function

$$\begin{aligned}V_{\mathcal{M}}(v,w,x,y,z;m) = &-Li_2\left(\frac{1}{vy}\right) - Li_2\left(\frac{1}{m^2wy}\right) + Li_2\left(\frac{w^2x}{y}\right)\\ &- Li_2\left(vm^2wy\right) - Li_2\left(vm^2wxz\right) - Li_2\left(m^2yz\right) + \frac{2\pi^2}{3}\\ &- \left(\log(vm^2)\right)^2 - 2\log v\log w - 2\log\left(m^2\right)\log(wy)\\ &- \log v\log(xy) - \log\left(m^2\right)\log z - \log y\log(zwy).\end{aligned}$$

- Hyperbolic volume

$$\mathrm{Im}\,V_{\mathcal{M}}(v,w,x,y,z;1) = -D\left(\frac{1}{vy}\right) - D\left(\frac{1}{wy}\right) + D\left(\frac{w^2x}{y}\right) - D(vwy) - D(vwxz) - D(yz)$$

$$\begin{aligned}V_{\mathcal{M}}(v,w,x,y,z;1) &= \frac{2\pi^2}{3} - L\left(\frac{1}{vy}\right) - L\left(\frac{1}{wy}\right) + L\left(\frac{w^2x}{y}\right)\\ &\quad - L(vwy) - L(vwxz) - L(yz) - \pi\mathrm{i}\log(vz)\\ &= 2\pi^2\cdot 0.10336\dots + \mathrm{i}\cdot 5.21257\dots\end{aligned}$$

with

$$\begin{pmatrix} v\\ w\\ x\\ y\\ z\end{pmatrix} = \begin{pmatrix} -1.1238 - 0.998279\mathrm{i}\\ -0.439261 - 0.570751\mathrm{i}\\ -0.836795 + 1.7323\mathrm{i}\\ -0.829546 - 0.0564355\mathrm{i}\\ -0.549394 + 0.740149\mathrm{i}\end{pmatrix}.$$

• A-polynomial

$$\begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -9 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 8 & -12 & -2 & 0 & 0 & 0 \\ 0 & 5 & -4 & 6 & -12 & -3 & 0 & 0 \\ 0 & -6 & 5 & -7 & 4 & -13 & -6 & -1 \\ 0 & 4 & -7 & 8 & -20 & 1 & -2 & 0 \\ 0 & -2 & 1 & -20 & 8 & -7 & 4 & 0 \\ -1 & -6 & -13 & 4 & -7 & 5 & -6 & 0 \\ 0 & 0 & -3 & -12 & 6 & -4 & 5 & 0 \\ 0 & 0 & 0 & -2 & -12 & 8 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -9 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

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